PART A

Unit - IV

Complex Variables - 2

4.1 Conformal Transformations

4.11 Introduction

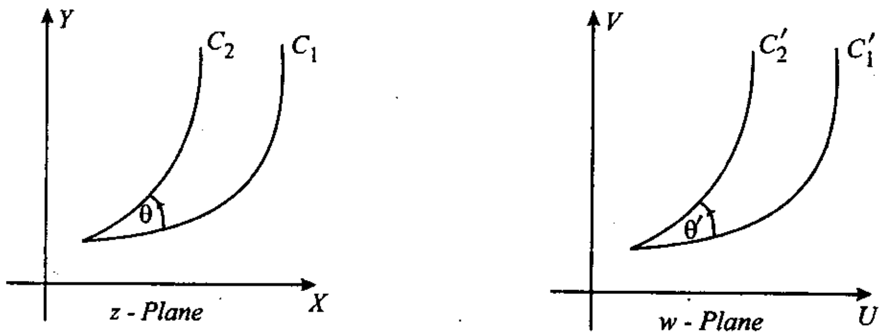
If $y = f(x)$ is a single valued function of the real variable x then for every value of x there corresponds a value y and the set of points (x, y) describe a curve C. This topic deals with the method of representing a complex valued function $w = f(z)$ geometrically.

4.12 Definitions and Theorem

Consider a complex valued function $w = f(z)$. Putting $z = x + iy$, $w = f(z) = u(x, y) + iv(x, y)$. The complex quantities $z = z(x, y)$, $w = w(u, v)$ are represented in two separate planes namely the z – plane and the w – plane respectively. A point (x, y) in the z – plane corresponds to a point (u, v) in the w – plane. If a set of points (x, y) traces a curve C in the z – plane and the corresponding points (u, v) traces a curve C' in the w – plane, we say that the curve C is ***transformed/mapped*** onto the curve C' under the transformation $w = f(z)$. The corresponding set of points in the two planes are called ***images*** of each other.

If a transformation preserves the angle between any two curves both in magnitude and sense then it is called a ***conformal transformation***. If only the magnitude of the angle is preserved then the transformation is called a ***Isogonal transformation.***

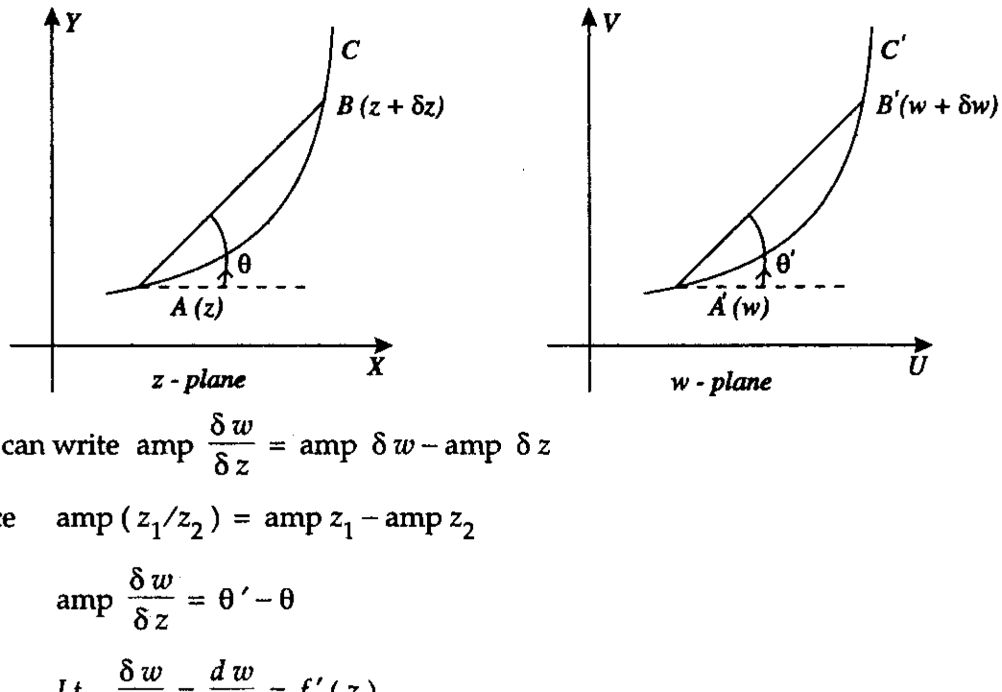
The transformation is conformal if $\theta = \theta'$ as in the following figure.



We now proceed to prove the sufficient condition for a complex valued function $w = f(z)$ to represent a conformal transformation.

Theorem : ***If $w = f(z)$ is an analytic function of z in a region of the z - plane then $w = f(z)$ is conformal at all points of the region where $f'(z) \neq 0$.***

Proof : Let $A(z)$, $B(z+\delta z)$ be the two neighbouring points on the curve C of the z - plane and $A'(w)$, $B'(w+\delta w)$ be the corresponding points on the curve C' of the w - plane.



We can write amp $\dfrac{\delta w}{\delta z}$ = amp δw − amp δz

since amp (z_1/z_2) = amp z_1 − amp z_2

ie., amp $\dfrac{\delta w}{\delta z} = \theta' - \theta$

$$\underset{\delta z \to 0}{Lt} \frac{\delta w}{\delta z} = \frac{dw}{dz} = f'(z).$$

Since $f'(z) \neq 0$ by hypothesis we can write,

$f'(z) = R e^{i\alpha}$ where $R \neq 0$. Taking amplitudes,

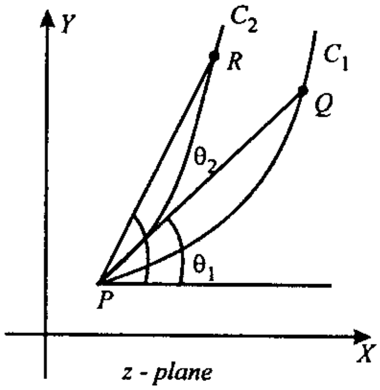
$\underset{\delta z \to 0}{Lt}$ amp $\dfrac{\delta w}{\delta z} = \alpha$ where α is a definite angle.

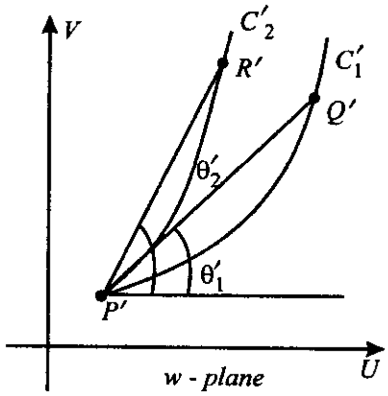
($\because$ *when $z = r e^{i\theta}$, r is called the modulus, θ is called the amplitude*)

ie., $\underset{B \to A}{Lt} (\theta' - \theta) = \alpha = \text{amp} f'(z)$ ...(1)

Let two curves C_1, C_2 intersect at a point $P(z_0)$ of the z - plane and the corresponding curves C'_1, C'_2 at a point $P'(w_0)$ of the w - plane.

Let Q, R ; Q', R' be the two neighbouring points of P and P' respectively.





From (1) $\theta_2' - \theta_2 = \alpha = \text{amp } f'(z_0)$

$\theta_1' - \theta_1 = \alpha = \text{amp } f'(z_0)$ in the limiting case.

Equating the LHS of these equaitions we have,

$$\theta_2' - \theta_2 = \theta_1' - \theta_1 \text{ or } \theta_2' - \theta_1' = \theta_2 - \theta_1$$

$$\therefore \quad Q'\hat{P}'R' = Q\hat{P}R$$

That is, angle of intersection at P is the same as the angle of intersection at P'.

Thus the transformation is conformal.

Note : *Since $f(z)$ is analytic, u and v satisfy $C-R$ equations $u_x = v_y$ and $v_x = -u_y$.*

We have the jacobian $J = \dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

ie., $\quad J = u_x v_y - u_y v_x = u_x \cdot u_x - (-v_x)v_x = u_x^2 + v_x^2$

But $\quad f'(z) = u_x + i v_x \qquad \therefore \quad |f'(z)|^2 = u_x^2 + v_x^2 = J$

This implies that for a conformal transformation the Jacobian J of u, v w.r.t. x, y must be different from zero.

4.2 Bilinear Transformation (BLT)

The transformation $w = \dfrac{az+b}{cz+d}$, where a, b, c, d are real / complex constants such that $ad - bc \neq 0$ is called a ***bilinear transformation.***

Remark. :

1. *The condition $ad - bc \neq 0$ ensures the conformal property of the BLT.* We have, $\dfrac{dw}{dz} = \dfrac{(cz+d)a - (az+b)c}{(cz+d)^2} = \dfrac{ad-bc}{(cz+d)^2}$, *$ad - bc \neq 0$ implies $\dfrac{dw}{dz} \neq 0$ and hence the transformation is conformal.*

2. *The cross ratio of a set of 4 points* (P_1, P_2, P_3, P_4) *is defined by*
$$\frac{(P_1 - P_2)(P_3 - P_4)}{(P_2 - P_3)(P_4 - P_1)}$$
3. ***Invariant points*** : *If a point z maps onto itself, that is* $w = z$ *under the bilinear transformation then the point is called an invariant point or a fixed point of the bilinear transformation.*
4. *Bilinear transformation is also called Mobius Transformation.*

Theorems on Bilinear Transformation

Theorem-1 ***If w is a bilinear transformation of z then z is a bilinear transformation of w. Also if z is a bilinear transformation of t then w is a bilinear transformation of t.***

Proof : By data $w = \dfrac{az + b}{cz + d}$ where $ad - bc \neq 0$

i.e., $w(cz + d) = az + b$

or $z(cw - a) = -dw + b \quad \therefore \quad z = \dfrac{-dw + b}{cw - a}$

Also $\dfrac{dz}{dw} = \dfrac{(cw - a)(-d) - (-dw + b)c}{(cw - a)^2} = \dfrac{ad - bc}{(cw - a)^2}$

Since $ad - bc \neq 0$ by data, **we conclude that** z **is a bilinear transformation of** w.

Further $z = \dfrac{-dw + b}{cw - a}$ is called the ***inverse bilinear transformation.***

Further if w is a bilinear transformation of z and z is a bilinear transformation of t, we shall prove that w is a bilinear transformation of t.

By data $w = \dfrac{az + b}{cz + d}$ where $ad - bc \neq 0$ ...(1)

$z = \dfrac{a_1 t + b_1}{c_1 t + d_1}$ where $a_1 d_1 - b_1 c_1 \neq 0$ ...(2)

Using (2) in the RHS of (1) we have,

$$w = \frac{a\left(\dfrac{a_1 t + b_1}{c_1 t + d_1}\right) + b}{c\left(\dfrac{a_1 t + b_1}{c_1 t + d_1}\right) + d} = \frac{aa_1 t + ab_1 + bc_1 t + bd_1}{a_1 ct + b_1 c + c_1 dt + dd_1}$$

i.e., $$w = \frac{(aa_1 + bc_1)t + (ab_1 + bd_1)}{(a_1c + c_1d)t + (b_1c + dd_1)} = \frac{At + B}{Ct + D} \quad \text{(say)}$$

where $A = aa_1 + bc_1$, $B = ab_1 + bd_1$

$C = a_1c + c_1d$, $D = b_1c + dd_1$

Now $AD - BC = (aa_1 + bc_1)(b_1c + dd_1) - (ab_1 + bd_1)(a_1c + c_1d)$

$$= (aca_1b_1 + ada_1d_1 + bcb_1c_1 + bdc_1d_1)$$

$$- (aca_1b_1 + adb_1c_1 + bca_1d_1 + bdc_1d_1)$$

$= ad(a_1d_1 - b_1c_1) - bc(a_1d_1 - b_1c_1)$ *(other terms cancelling)*

$AD - BC = (a_1d_1 - b_1c_1)(ad - bc) \neq 0$ by using the data.

Thus $w = \dfrac{At + B}{Ct + D}$ where $AD - BC \neq 0$

This proves that w is a bilinear transformation of t.

Theorem-2 ***The cross ratio of a set of four points is preserved (remain invariant) under a bilinear transformation.***

Proof : Let $w = \dfrac{az + b}{cz + d}$ where $ad - bc \neq 0$ be the bilinear transformation and let w_1, w_2, w_3, w_4 be the images of z_1, z_2, z_3, z_4 under this bilinear transformation.

We have to prove that,

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \quad \ldots(1)$$

Now, $w_1 - w_2 = \dfrac{az_1 + b}{cz_1 + d} - \dfrac{az_2 + b}{cz_2 + d}$

$$= \frac{(az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d)}{(cz_1 + d)(cz_2 + d)}$$

$$= \frac{(acz_1z_2 + adz_1 + bcz_2 + bd) - (acz_1z_2 + adz_2 + bcz_1 + bd)}{(cz_1 + d)(cz_2 + d)}$$

$$(w_1 - w_2) = \frac{ad(z_1 - z_2) - bc(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}$$

$$\therefore \quad (w_1 - w_2) = \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}$$

By symmetry we can simply write down the expression for $(w_3 - w_4)$, $(w_2 - w_3)$ and $(w_4 - w_1)$

LHS of (1) becomes,

$$\frac{\dfrac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \cdot \dfrac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)}}{\dfrac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)} \cdot \dfrac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)}} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} = \text{RHS}$$

This proves that the cross ratio of a set of four points is preserved under the bilinear transformation.

<u>Theorem-3</u> ***Every bilinear transformation map circles or straight lines in the z-plane into circles or straight lines in the w-plane.***

Note : *The general equation of a circle in the cartesian form* $x^2 + y^2 + 2gx + 2fy + c = 0$ *can be put in the complex form :*

$Az\bar{z} + B\bar{z} + \bar{B}z + C = 0$ *where* A, C *are real constants and* B *is a complex constant satisfying the condition* $B\bar{B} \geq AC$. *Further when* $A = 0$, *the equation represents a straight line.*

Proof Let $w = \dfrac{az + b}{cz + d}$ where $ad - bc \neq 0$ be the BLT.

ie., $\quad w(cz + d) = az + b$ or $z(cw - a) = -dw + b$

$$\therefore \quad z = \frac{-dw + b}{cw - a}$$

This gives $\bar{z} = \dfrac{-\bar{d}\,\bar{w} + \bar{b}}{\bar{c}\,\bar{w} - \bar{a}}$ where $\bar{a}\bar{d} - \bar{b}\bar{c} \neq 0$

Let us consider the equation of a circle in the complex form,

$$Az\bar{z} + B\bar{z} + \bar{B}z + C = 0 \qquad \ldots(1)$$

Substituting the expressions for z, $\bar{z}$ we obtain,

$$A\left(\frac{-dw+b}{cw-a}\right)\left(\frac{-\bar{d}\bar{w}+\bar{b}}{\bar{c}\bar{w}-\bar{a}}\right)+B\left(\frac{-\bar{d}\bar{w}+\bar{b}}{\bar{c}\bar{w}-\bar{a}}\right)+\bar{B}\left(\frac{-dw+b}{cw-a}\right)+C=0$$

i.e., $A(-dw+b)(-\bar{d}\bar{w}+\bar{b})+B(cw-a)(-\bar{d}\bar{w}+\bar{b})$

$$+\bar{B}(-dw+b)(\bar{c}\bar{w}-\bar{a})+C(cw-a)(\bar{c}\bar{w}-\bar{a})=0$$

Multiplying and collecting the coefficient of the like terms, this equation can be put in the form

$$Pw\bar{w}+Q\bar{w}+Rw+S=0 \qquad \ldots(2)$$

where $P=(Ad\bar{d}-Bc\bar{d}-\bar{B}\bar{c}d+Cc\bar{c})$

$Q=(-Ab\bar{d}+Ba\bar{d}+\bar{B}b\bar{c}-Ca\bar{c})$

$R=(-A\bar{b}d+B\bar{b}c+\bar{B}\bar{a}d-C\bar{a}c)$

$S=(Ab\bar{b}-Ba\bar{b}-\bar{B}b\bar{a}+Ca\bar{a})$

It may be observed that $R=\bar{Q}$ and hence (2) assumes the form

$$Pw\bar{w}+Q\bar{w}+\bar{Q}w+S=0 \qquad \ldots(3)$$

Further $Q\bar{Q}-PS$ can be simplified into the form

$(B\bar{B}-AC)(ad-bc)(\bar{a}\bar{d}-\bar{b}\bar{c})=(B\bar{B}-AC)\,|\,ad-bc\,|^2$

($\because$ $\alpha\bar{\alpha}=|\,\alpha\,|^2$ *for any complex constant* α)

We conclude that $Q\bar{Q}-PS\geq 0$ as we have $B\bar{B}\geq AC$ by data.

$\therefore \quad Q\bar{Q}\geq PS$

Hence we conclude that equation (3) represents a circle. It is obvious that a straight line is transformed into a straingt line as a straight line can be regarded as a circle of infinite radius. (*It can be worked independently also*).

Thus we have proved that every bilinear transformation map circles or straight lines in the z- plane into circles or straight lines in the w-plane.

WORKED PROBLEMS

Finding the bilinear transformation given the image of a set of three points.

Working procedure for problems

- Given w_1, w_2, w_3 corresponding to z_1, z_2, z_3, we assume the bilinear transformation in the form $w=\dfrac{az+b}{cz+d}$
- We substitute the given set of points to obtain a set of three equations in four unknowns a, b, c, d.
- We deduce a pair of equation in any three unknowns and solve by the rule of cross multiplication to obtain a proportionate set of values for the three unknowns.

➲ These values are used to find the fourth unknown.

➲ All these four values when substituted in the assumed form of w will give us the required bilinear transformation.

1. *Find the bilinear transformation which map the points* $z = 1, i, -1$ *into* $w = i, 0, -i$. *Under this transformation find the image of* $|z| < 1$

>> Let $w = \dfrac{az + b}{cz + d}$ be the required bilinear transformation.

We shall substitute the given values of z and w to obtain three equations as follows.

$$z = 1,\ w = i \text{ gives } i = \frac{a+b}{c+d}$$

i.e., $a + b - ci - di = 0$ ...(1)

$$z = i,\ w = 0 \text{ gives } 0 = \frac{ai + b}{ci + d}$$

i.e., $ai + b = 0$ ...(2)

$$z = -1,\ w = -i \text{ gives } -i = \frac{-a+b}{-c+d}$$

i.e., $-a + b - ci + di = 0$ ...(3)

Now (1) + (3) gives $2b - 2ci = 0$

or $b - ci = 0$ ...(4)

We shall solve (2) and (4) by writing them in the form

$ia + 1b + 0c = 0$ ...(2)

$0a + 1b - ic = 0$ ...(4)

Apply ; the rule of cross multiplication we have,

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix}} = \frac{c}{\begin{vmatrix} i & 1 \\ 0 & 1 \end{vmatrix}}$$

i.e., $\dfrac{a}{-i} = \dfrac{-b}{-i^2} = \dfrac{c}{i}$ or $\dfrac{a}{-i} = \dfrac{b}{-1} = \dfrac{c}{i} = k$ (*say*)

$\therefore$ $a = -ik,\ b = -k,\ c = ik.$

Substituting these in (1), we have

$-ik - k + k - di = 0$ *i.e.,* $-(di + ik) = 0$ or $d = -k$

Substituting the values of a, b, c, d in the assumed bilinear transformation we have,

$$w = \frac{-ikz - k}{ikz - k} = \frac{-k(1+iz)}{-k(1-iz)}$$

Thus $w = \dfrac{1+iz}{1-iz}$ is the required bilinear transformation.

Remark : *Alternative method to find a , b , c , d is presented in some of the problems to follow .*

Note :

1. *The answer can be verified by substituting the values of z in the RHS and the resulting w must tally with that of the data.*

If $z = 1,\ w = \dfrac{1+i}{1-i} = \dfrac{(1+i)^2}{1-i^2} = \dfrac{1+i^2+2i}{1-(-1)} = \dfrac{2i}{2} = i$

If $z = i,\ w = \dfrac{1+i^2}{1-i^2} = \dfrac{1-1}{1+1} = \dfrac{0}{2} = 0$

If $z = -1,\ w = \dfrac{1-i}{1+i} = \dfrac{(1-i)^2}{1-i^2} = \dfrac{1+i^2-2i}{2} = -i$

2. *Since k is a constant multiple in the values of a, b, c, d it gets cancelled in the final form of w. Therefore we can as well avoid this constant multiple in the values of a, b, c, d.*

Now let us find the image of $|z| < 1$ under the obtained BLT.

The bilinear transformation is $w = \dfrac{1+iz}{1-iz}$

ie., $1+iz = w(1-iz)$ or $z(i+iw) = w-1$ or $z = \dfrac{1}{i}\left(\dfrac{w-1}{w+1}\right) = i\left(\dfrac{1-w}{1+w}\right)$

$$|z| < 1 \Rightarrow |i| \left|\frac{1-w}{1+w}\right| < 1 \text{ or } |1-w|^2 < |1+w|^2$$

ie., $|1-(u+iv)|^2 < |1+(u+iv)|^2$

ie., $|(1-u)-iv|^2 < |(1+u)+iv|^2$

ie., $(1-u)^2+v^2 < (1+u)^2+v^2$

ie., $-2u < 2u$ or $0 < 4u$ or $4u > 0 \Rightarrow u > 0$

Thus $u > 0$ is the image of $|z| < 1$

2. *Find the bilinear transformation which map the points : $z = 1,\ i, -1$ into $w = 2,\ i, -2$. Also find the invariant points of the transformation.*

>> Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

$$z = 1,\ w = 2\ ;\ 2 = \frac{a+b}{c+d}$$

i.e., $a + b - 2c - 2d = 0$ ... (1)

$$z = i,\ w = i\ ;\ i = \frac{ai+b}{ci+d}$$

i.e., $ai + b + c - di = 0$ ... (2)

$$z = -1,\ w = -2\ ;\ -2 = \frac{-a+b}{-c+d}$$

i.e., $-a + b - 2c + 2d = 0$ ... (3)

(1) + (3) gives $2b - 4c = 0$

or $b - 2c = 0$ ... (4)

(2) + $i \times$ (3) gives,

$(1+i)b + (1-2i)c + id = 0$ ... (5)

Let us solve (4) and (5) by writing them in the form,

$1b - 2c + 0d = 0$ ... (4)

$(1+i)b + (1-2i)c + id = 0$ ... (5)

Applying the rule of cross multiplication we have,

$$\frac{b}{\begin{vmatrix} -2 & 0 \\ (1-2i) & i \end{vmatrix}} = \frac{-c}{\begin{vmatrix} 1 & 0 \\ (1+i) & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 1 & -2 \\ (1+i) & (1-2i) \end{vmatrix}}$$

i.e., $\dfrac{b}{-2i} = \dfrac{-c}{i} = \dfrac{d}{3}$

$\therefore$ $b = -2i,\ c = -i,\ d = 3.$

With these values, (1) becomes,

$a - 2i + 2i - 6 = 0 \quad \therefore \quad a = 6$

Thus by substituting the values of a, b, c, d the required bilinear transformation is

$$w = \frac{6z - 2i}{-iz + 3}$$

Further, the invariant points of this transformation are obtained by taking $w = z$

i.e., $\quad z = \dfrac{6z - 2i}{-iz + 3}$

i.e., $\quad -iz^2 + 3z - 6z + 2i = 0$

i.e., $\quad -iz^2 - 3z + 2i = 0$

Applying the quadratic formula we have,

$$z = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-i)(2i)}}{-2i} = \frac{3 \pm \sqrt{9-8}}{-2i}$$

i.e., $\quad z = \dfrac{3 \pm 1}{-2i} = \dfrac{4}{-2i}, \ \dfrac{2}{-2i} = \dfrac{-2}{i}, \ \dfrac{-1}{i} = 2i, \ i$

Thus $\;$ **$z = 2i, \ i$ are the invariant points.**

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3. *Find the bilinear transformation which maps* $z_1 = -1, \ z_2 = 0, \ z_3 = 1$ *into* $w_1 = 0, \ w_2 = i, \ w_3 = 3i$.

>> Let $w = \dfrac{az + b}{cz + d}$ be the required bilinear transformation.

$$z_1 = -1, \ w_1 = 0 \ ; \ 0 = \frac{-a+b}{-c+d}$$

i.e., $\quad -a + b = 0$ $\qquad \ldots (1)$

$$z_2 = 0, \ w_2 = i \ ; \ i = \frac{0 + b}{0 + d}$$

i.e., $\quad b - di = 0$ $\qquad \ldots (2)$

$$z_3 = 1, \ w_3 = 3i \ ; \ 3i = \frac{a + b}{c + d}$$

i.e., $\quad a + b - 3ic - 3id = 0$ $\qquad \ldots (3)$

Now $\quad$ (1) – (2) gives,

$-a + di = 0$ $\qquad \ldots (4)$

Let us solve (2) and (4) by writing them in the form

$0a + 1b - id = 0$ $\qquad \ldots (2)$

$-1a + 0b + id = 0$ $\qquad \ldots (4)$

Applying the rule of cross multiplication we have,

$$\frac{a}{\begin{vmatrix} 1 & -i \\ 0 & i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & -i \\ -1 & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}}$$

i.e., $\quad \frac{a}{i} = \frac{-b}{-i} = \frac{d}{1}$

$\therefore \quad a = i, \quad b = i, \quad d = 1$

With these values (3) becomes $i + i - 3ic - 3i = 0$

i.e., $\quad -3ic - i = 0$ or $3c + 1 = 0 \quad \therefore \quad c = -1/3$

Substituting the values of a, b, c, d the assumed bilinear transformation becomes

$$w = \frac{iz + i}{-z/3 + 1} = \frac{3i(z+1)}{-z+3}$$

This can also be written in the form $w = \dfrac{3i(z+1)}{i(iz - 3i)}$

Thus the required bilinear transformation is $w = \dfrac{3z + 3}{iz - 3i}$

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4. *Find the bilinear transformation which maps $z = \infty, i, 0$ into $w = -1, -i, 1$. Also find the fixed points of the transformation.*

>> Let $w = \dfrac{az + b}{cz + d}$ be the required bilinear transformation.

$z = \infty$, $w = -1$; the bilinear transformation is to be written in the form

$$w = \frac{z[a + (b/z)]}{z[c + (d/z)]} = \frac{a + (b/z)}{c + (d/z)}$$

$\therefore \quad -1 = \dfrac{a+0}{c+0} \quad (\because 1/z = 0 \text{ when } z = \infty)$

i.e., $\quad a + c = 0 \qquad \ldots (1)$

$$z = i, \; w = -i \; ; \; -i = \frac{ai + b}{ci + d}$$

i.e., $\quad ai + b - c + di = 0 \qquad \ldots (2)$

$$z = 0, \; w = 1 \; ; \; 1 = \frac{0 + b}{0 + d}$$

i.e., $\quad b - d = 0 \qquad \ldots (3)$

Now (1) + (2) gives,

$$(1+i)a + b + id = 0 \qquad \ldots(4)$$

Let us solve (3) and (4) by writing them in the form

$$0a \quad + 1b - 1d = 0 \qquad \ldots(3)$$

$$(1+i)a + 1b + id = 0 \qquad \ldots(4)$$

Applying the rule of cross multiplication we have,

$$\frac{a}{\begin{vmatrix} 1 & -1 \\ 1 & i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & -1 \\ 1+i & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 0 & 1 \\ 1+i & 1 \end{vmatrix}}$$

$$\frac{a}{i+1} = \frac{-b}{1+i} = \frac{d}{-(1+i)} \quad \text{or} \quad \frac{a}{1} = \frac{b}{-1} = \frac{d}{-1}$$

$\therefore \qquad a = 1,\ b = -1,\ d = -1$

Also from (1) $\quad c = -a \quad \therefore \quad c = -1$

Substituting the values of a, b, c, d the assumed bilinear transformation becomes

$$w = \frac{1.z - 1}{-1.z - 1}$$

Thus $w = \dfrac{1-z}{1+z}$ is the required bilinear transformation.

Further, the invariant points are obtained by taking $w = z$

i.e., $\qquad z = \dfrac{1-z}{1+z} \quad \text{or} \quad z + z^2 = 1 - z$

i.e., $\qquad z^2 + 2z - 1 = 0$

$$\therefore \qquad z = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

Thus the invariant points are $-1 + \sqrt{2}$ and $-1 - \sqrt{2}$.

Note : *If the equations are simple we can find a, b, c, d without going to the method of cross multiplication and the same is illustrated as follows.*

Using $c = -a$ *and* $d = b$ *in (2) we have* $ai + b + a + bi = 0$

or $\qquad a(1+i) = -b(1+i) \Rightarrow a = -b$

So we have $a = -b,\ c = b,\ d = b$. *Choosing* $b = 1$ *for convenience we have* $a = -1,\ c = 1,\ d = 1$.

The required BLT is $w = \dfrac{-z+1}{z+1}$ or $w = \dfrac{1-z}{1+z}$

[This technique can be employed in the earlier problems also]

5. *Find the bilinear transformation which maps the points $z = 1, i, -1$ into $w = 0, 1, \infty$*

>> Let $w = \dfrac{az + b}{cz + d}$ be the required bilinear transformation.

$$z = 1,\ w = 0\ ;\quad 0 = \frac{a+b}{c+d}$$

i.e., $\quad a + b = 0 \qquad \ldots(1)$

$$z = i,\ w = 1\ ;\quad 1 = \frac{ai+b}{ci+d}$$

i.e., $\quad ai + b - ci - d = 0 \qquad \ldots(2)$

$$z = -1,\ w = \infty\ ;\ \text{Consider } \frac{1}{w} = \frac{cz + d}{az + b}$$

When $z = -1,\ w = \infty$; we have $1/w = 0$

$$\therefore \quad 0 = \frac{-c + d}{-a + b}$$

i.e., $\quad -c + d = 0 \qquad \ldots(3)$

Now (2) + (3) gives,

$$ai + b - (1+i)c = 0 \qquad \ldots(4)$$

Let us solve (1) and (4) by writing them in the form

$$1a + 1b + 0c = 0 \qquad \ldots(1)$$

$$ia + 1b - (1+i)c = 0 \qquad \ldots(4)$$

Applying the rule of cross multiplication we have,

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -(1+i) \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & 0 \\ i & -(1+i) \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 1 \\ i & 1 \end{vmatrix}}$$

i.e., $$\frac{a}{-(1+i)} = \frac{-b}{-(1+i)} = \frac{c}{1-i}$$

$$a = -(1+i),\ b = (1+i),\ c = (1-i).\ \text{ Also } d = c = (1-i)$$

Substituting these values in the assumed BLT we have,

$$w = \frac{-(1+i)z + (1+i)}{(1-i)z + (1-i)}$$

i.e., $w = \dfrac{(1+i)}{(1-i)}\left(\dfrac{1-z}{1+z}\right)$

Multiplying and dividing with $(1+i)$ we obtain

$$w = \frac{(1+i)^2}{1-i^2}\left(\frac{1-z}{1+z}\right) = \frac{1+i^2+2i}{2}\left(\frac{1-z}{1+z}\right) = i\left(\frac{1-z}{1+z}\right)$$

Thus the required bilinear transformation is $w = i\left(\dfrac{1-z}{1+z}\right)$

---

6. *Find the bilinear transformation that map the points : $z_1 = 0,\ z_2 = -i,\ z_3 = 2i$ into the points $w_1 = 5i,\ w_2 = \infty,\ w_3 = -i/3$ respectively. What are the invariant points of the transformation ?*

>> Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

$z_1 = 0,\ w_1 = 5i\ ;\quad 5i = \dfrac{b}{d}$

i.e., $b - 5id = 0$ ...(1)

$z_2 = -i,\ w_2 = \infty\ ;$ Consider $\dfrac{1}{w} = \dfrac{cz+d}{az+b}$

$\therefore\quad 0 = \dfrac{-ci+d}{-ai+b}$

i.e., $-ci + d = 0$ ...(2)

$z_3 = 2i,\ w_3 = -i/3\ ;\quad \dfrac{-i}{3} = \dfrac{2ia+b}{2ic+d}$

i.e., $6ia + 3b - 2c + id = 0$

Let us solve (1) and (2) by writing them in the form

$1b + 0c - 5id = 0$ ...(1)

$0b - ic + 1d = 0$ ...(2)

Applying the rule of cross multiplication we have,

$$\frac{b}{\begin{vmatrix} 0 & -5i \\ -i & 1 \end{vmatrix}} = \frac{-c}{\begin{vmatrix} 1 & -5i \\ 0 & 1 \end{vmatrix}} = \frac{d}{\begin{vmatrix} 1 & 0 \\ 0 & -i \end{vmatrix}}$$

i.e., $\dfrac{b}{5} = \dfrac{-c}{1} = \dfrac{d}{-i}$

$\therefore \quad b = 5,\ c = -1,\ d = -i$

Substituting these values in (3) we have,

$$6ia + 15 + 2 + 1 = 0 \text{ or } 6ia = -18 \quad \therefore \quad a = -3/i = 3i$$

Thus the required bilinear transformation is

$$w = \frac{3iz + 5}{-z - i} = \frac{-3z + 5i}{-iz + 1}$$

Note : *$b = 5id$ and $c = d/i = -id$ from (1) and (2) respectively.*

Using these in (3) we get $a = -3d$

Choosing $d = 1$, we have $b = 5i$, $c = -i$ and $a = -3$

Thus the required BLT is $w = \dfrac{-3z + 5i}{-iz + 1}$

Now, let us take $w = z$ for finding the invariant points.

i.e., $\quad z = \dfrac{3iz + 5}{-z - i}$ or $-z^2 - iz - 3iz - 5 = 0$

Hence we have, $\quad z^2 + 4iz + 5 = 0$

$$\therefore \quad z = \frac{-4i \pm \sqrt{(4i)^2 - 4.1.5}}{2} = -\frac{4i \pm \sqrt{36i^2}}{2} = \frac{-4i \pm 6i}{2}$$

i.e., $\quad z = \dfrac{-4i + 6i}{2},\ \dfrac{-4i - 6i}{2}$ or $z = i,\ -5i$

Thus the invariant points are $z = i,\ -5i$

---

7. *Find the bilinear transformation which map the points $z = 0, 1, \infty$ into the points $w = -5, -1, 3$ respectively. What are the invariant points in this transformation?*

>> Let $w = \dfrac{az + b}{cz + d}$ be the required bilinear transformation.

$$z = 0,\ w = -5\ ;\ -5 = \frac{b}{d}$$

ie., $\quad b = -5d$ $\quad \ldots (1)$

$$z = 1,\ w = -1\ ;\ -1 = \frac{a + b}{c + d}$$

ie., $\quad a + b + c + d = 0$ $\quad \ldots (2)$

$$z = \infty,\ w = 3\ ;\ w = \frac{z[a + (b/z)]}{z[c + (d/z)]} = \frac{a + (b/z)}{c + (d/z)}$$

Now consider $z = \infty,\ w = 3\ ;\ \ 3 = \dfrac{a+0}{c+0}$

ie., $\quad a = 3c$ . . . (3)

Using (1) and (3) in (2) we get $4c - 4d = 0$ or $c = d$

Choosing $d = 1$ we get $c = 1,\ b = -5,\ a = 3.$

Thus the required bilinear transformation is $w = \dfrac{3z-5}{z+1}$

The invariant points are obtained by taking $w = z$

ie., $\quad z = \dfrac{3z-5}{z+1}$ or $z^2 + z = 3z - 5$ or $z^2 - 2z + 5 = 0$

Hence $z = \dfrac{2 \pm \sqrt{4-20}}{2} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

Thus $z = 1 \pm 2i$ **are the invariant points.**

Miscellaneous problems

8. *Find the map of the real axis of the z-plane in the w-plane under the transformation* $w = \dfrac{1}{z+i}$

>> The equation of the real axis of the z-plane is $y = 0$ and we have by data,

$$w = \frac{1}{z+i} \quad \text{or} \quad z + i = \frac{1}{w} \quad \text{or} \quad z = \frac{1}{w} - i$$

Hence we have,

$$x + iy = \frac{1}{u+iv} - i = \frac{(u-iv)}{(u+iv)(u-iv)} - i = \frac{u - iv}{u^2 + v^2} - i$$

i.e., $$x + iy = \frac{u}{u^2 + v^2} + i\left(\frac{-v}{u^2 + v^2} - 1\right)$$

Equating the imaginary parts we get $y = \dfrac{-v}{u^2 + v^2} - 1$

Putting $y = 0$ (*the equations of the real axis*) we have , $\dfrac{-v}{u^2 + v^2} - 1 = 0$

ie., $\quad -u^2 - v^2 - v = 0 \quad$ or $u^2 + v^2 + v = 0$

i.e., $$(u - 0)^2 + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

or $$(u - 0)^2 + \left\{v - \left(-\frac{1}{2}\right)\right\}^2 = \left(\frac{1}{2}\right)^2$$

This is a circle in the w-plane with centre $(0, -1/2)$ and radius $1/2$.

Thus we conclude that **the map of the real axis of the z-plane is a circle in the w-plane.**

---

9. *Show that the transformation* $w = \dfrac{2z+3}{z-4}$ *maps the circle* $x^2 + y^2 - 4x = 0$ *into a straight line.*

>> By data $w = \dfrac{2z+3}{z-4}$

ie., $w(z-4) = 2z+3$ or $z(w-2) = 4w+3$

$$\therefore \quad z = \frac{4w+3}{w-2} \qquad \ldots (1)$$

Since $z = x + iy$; $\bar{z} = x - iy$, $z\bar{z} = x^2 + y^2$ and $\dfrac{z+\bar{z}}{2} = x$

Using these in $x^2 + y^2 - 4x = 0$ we have,

$$z\bar{z} - 2(z+\bar{z}) = 0 \qquad \ldots (2)$$

Also we have from (1) $z = \dfrac{4w+3}{w-2}$ and hence $\bar{z} = \dfrac{4\bar{w}+3}{\bar{w}-2}$

Using these in (2) we have,

$$\left(\frac{4w+3}{w-2}\right)\left(\frac{4\bar{w}+3}{\bar{w}-2}\right) - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

i.e., $(4w+3)(4\bar{w}+3) - 2(4w+3)(\bar{w}-2) - 2(4\bar{w}+3)(w-2) = 0$

ie., $(16w\bar{w} + 12\bar{w} + 12w + 9) - (8w\bar{w} - 16w + 6\bar{w} - 12)$
$- (8w\bar{w} - 16\bar{w} + 6w - 12) = 0$

On simplification we obtain, $22\bar{w} + 22w + 33 = 0$

Dividing by 11, we have, $2(\bar{w} + w) + 3 = 0$

But $w = u + iv$; $\bar{w} = u - iv$ and we have $\bar{w} + w = 2u$

Thus we get $4u + 3 = 0$ which is a straight line in the w-plane.

---

10. *Show that the transformation* $w = \dfrac{i-z}{i+z}$ *maps the x-axis of the z-plane onto a circle* $|w| = 1$ *and the points in the half plane* $y > 0$ *onto the points* $|w| < 1$.

>> By data $w = \dfrac{i-z}{i+z}$

ie., $w(i+z) = i-z$ or $z(w+1) = i - iw$

$\therefore \quad z = \dfrac{i(1-w)}{(1+w)}$

i.e., $\quad x + iy = \dfrac{i(1-u-iv)}{1+u+iv} = \dfrac{v + i(1-u)}{(1+u) + iv}$

i.e., $\quad x + iy = \dfrac{v + i(1-u)}{(1+u) + iv} \cdot \dfrac{(1+u) - iv}{(1+u) - iv}$

$$= \frac{[v(1+u) - i^2 v(1-u)] + i[(1-u)(1+u) - v^2]}{(1+u)^2 - i^2 v^2}$$

i.e., $\quad x + iy = \left[\dfrac{2v}{(1+u)^2 + v^2}\right] + i\left[\dfrac{1-u^2-v^2}{(1+u)^2 + v^2}\right]$

y is obtained by equating the imaginary parts on both sides.

Since $y = 0$ is the equation of the x-axis we have,

$$\frac{1-u^2-v^2}{(1+u)^2+v^2} = 0 \quad \text{or} \quad 1 - u^2 - v^2 = 0$$

$\therefore$ **$u^2 + v^2 = 1$ which is the circle $|w|^2 = 1$ or $|w| = 1$**

Further $y > 0 \Rightarrow 1 - u^2 - v^2 > 0 \Rightarrow 1 > u^2 + v^2$

i.e., $\quad u^2 + v^2 < 1$ or $|w| < 1$

Thus we have proved the desired results.

---

11. *Given* $w = \dfrac{iz+2}{4z+i}$, *find (a) the inverse of the given transformation.*

(b) the centre and radius of the circle mapped by the real axis of the z-plane.

>> By data, $w = \dfrac{iz+2}{4z+i}$

i.e., $\quad w(4z + i) = iz + 2$ or $z(4w - i) = 2 - iw$

Thus $\quad z = \dfrac{-iw+2}{4w-i}$ is the required inverse transformation.

Now using $z = x + iy$ and $w = u + iv$ we have

$$x + iy = \frac{-i(u+iv) + 2}{4(u+iv) - i} = \frac{(v+2) - iu}{4u + i(4v-1)}$$

$$x + iy = \frac{(v+2) - iu}{4u + i(4v-1)} \cdot \frac{4u - i(4v-1)}{4u - i(4v-1)}$$

$$x + iy = \frac{[\,4u\,(v+2) - u\,(4v-1)\,] - i\,[\,4u^2 + (4v-1)(v+2)\,]}{16u^2 + (4v-1)^2}$$

$$x + iy = \left[\frac{9u}{16u^2 + (4v-1)^2}\right] - i\left[\frac{4u^2 + 4v^2 + 7v - 2}{16u^2 + (4v-1)^2}\right]$$

Since the equation of the real axis (*x-axis*) of the z-plane is $y = 0$, equating the imaginary parts on both sides we get y and hence we have

$$-\frac{4u^2 + 4v^2 + 7v - 2}{16u^2 + (4v-1)^2} = 0 \quad \text{or} \quad 4u^2 + 4v^2 + 7v - 2 = 0$$

Dividing by 4 we get, $u^2 + v^2 + \frac{7}{4}v - \frac{1}{2} = 0$ which is a circle.

This can be written in the form,

$$(u-0)^2 + \left(v + \frac{7}{8}\right)^2 - \frac{49}{64} - \frac{1}{2} = 0$$

i.e., $$(u-0)^2 + \left\{v - \left(-\frac{7}{8}\right)\right\}^2 = \frac{81}{64} = \left(\frac{9}{8}\right)^2$$

Thus the centre of the circle is $(0, -7/8)$ and radius is $9/8$

---

12. *Show that the transformation $w = i\left(\frac{1-z}{1+z}\right)$ transforms the circle with centre origin and unit radius of the z-plane into the real axis of the w-plane. Also show that the interior of the circle is mapped onto the upper half of the w-plane.*

>> The equation of the circle in the z-plane is $x^2 + y^2 = 1$

ie., $|z|^2 = 1$ or $|z| = 1$ and the interior of the circle is $|z| < 1$

$\therefore$ we have to find the image in the w-plane corresponding to $|z| \le 1$

By data $w = i\left(\frac{1-z}{1+z}\right)$

i.e., $w\,(1+z) = i - iz$ or $z\,(w+i) = i - w$

$\therefore$ $z = \frac{i-w}{i+w}$ and $\bar{z} = \frac{-i-\bar{w}}{-i+\bar{w}}$.

Also we have $z\bar{z} = |z|^2$

Consider $z\bar{z} = \left(\dfrac{i-w}{i+w}\right)\left(\dfrac{-i-\bar{w}}{-i+\bar{w}}\right) = \dfrac{(1+w\bar{w}) + i(w-\bar{w})}{(1+w\bar{w}) - i(w-\bar{w})}$

$z\bar{z} \le 1$ or $|z|^2 \le 1 \Rightarrow |z| \le 1.$

Hence we have,

$$(1 + w\bar{w}) + i(w - \bar{w}) \le (1+w\bar{w}) - i(w - \bar{w})$$

i.e., $2i(w - \bar{w}) \le 0.$

But $w - \bar{w} = (u+iv) - (u-iv) = 2iv$

Hence $2i(2iv) \le 0$ or $-4v \le 0$ or $v \ge 0$

$v = 0$ is the equation of the real axis in the w - plane.

Thus we conclude that the boundary of the circle $|z| = 1$ is mapped onto the real axis and the interior $|z| < 1$ is mapped onto $v > 0$ being the upper half of the w-plane.

This proves the desired result.

---

13. *Show that the transformation* $z = \dfrac{2}{w+1}$ *maps the region D of the z-plane bounded by the two circles* $x^2 + y^2 + 2y = 0$ *and* $x^2 + y^2 + y = 0$ *conformally onto the strip of the w-plane bounded by the lines* $v = 1$ *and* $v = 2$.

>> By data $z = \dfrac{2}{w+1}$ and hence $\bar{z} = \dfrac{2}{\bar{w}+1}$

Also $z = x+iy, \quad \bar{z} = x-iy, \quad z\bar{z} = x^2 + y^2, \quad \dfrac{z-\bar{z}}{2i} = y$

Consider $x^2 + y^2 + 2y = 0$

i.e., $z\bar{z} + \dfrac{1}{i}(z-\bar{z}) = 0.$

Substituting for z and $\bar{z}$ we have,

$$\frac{2}{w+1} \cdot \frac{2}{\bar{w}+1} + \frac{1}{i}\left(\frac{2}{w+1} - \frac{2}{\bar{w}+1}\right) = 0$$

i.e., $4 + \dfrac{2}{i}(\bar{w} + 1 - w - 1) = 0$

i.e., $4 + \dfrac{2}{i}(\bar{w} - w) = 0$

i.e., $4 + \dfrac{2}{i}(u-iv-u-iv) = 0$ or $4 + \dfrac{2}{i}(-2iv) = 0$

i.e., $4 - 4v = 0$ or $v = 1$

Again consider $x^2 + y^2 + y = 0$

i.e., $z\bar{z} + \frac{1}{2i}(z - \bar{z}) = 0$

i.e., $\frac{2}{w+1} \cdot \frac{2}{\bar{w}+1} + \frac{1}{2i}\left(\frac{2}{w+1} - \frac{2}{\bar{w}+1}\right) = 0$

i.e., $4 + \frac{1}{i}(\bar{w} + 1 - w - 1) = 0$

i.e., $4 + \frac{1}{i}(-2iv) = 0$ or $4 - 2v = 0$ or $v = 2$.

This proves the desired result.

4.3 Discussion of Conformal Transformations

Given the transformation $w = f(z)$, we put $z = x + iy$ or $z = re^{i\theta}$ to obtain u and v as functions of x, y or r, θ. We find the image in w – plane corresponding to the given curve in the z – plane. Sometimes we need to make some judicious elimination from u and v for obtaining the image in the w – plane.

At the end of the discussion of every transformation, a few possible questions on the transformation is given for the benefit of the reader.

4.31 Discussion of $w = z^2$

Consider $w = z^2$

ie., $u + iv = (x + iy)^2$ or $u + iv = (x^2 - y^2) + i(2xy)$

$\therefore$ $u = x^2 - y^2$ and $v = 2xy$ ... (1)

Case-1 : Let us consider $x = c_1$, c_1 is a constant.

The set of equations (1) become $u = c_1^2 - y^2$; $v = 2c_1 y$

Now $y = v/2c_1$ and substituting this in u we get

$$u = c_1^2 - (v^2/4c_1^2) \text{ or } v^2/4c_1^2 = c_1^2 - u \text{ or } v^2 = -4c_1^2(u - c_1^2)$$

This is a parabola in the w -plane symmetrical about the real axis with its vertex at $(c_1^2, 0)$ and focus at the origin. It may be observed that the line $x = -c_1$ is also transformed into the same parabola.

Case-2 : Let us consider $y = c_2$, c_2 is a constant.

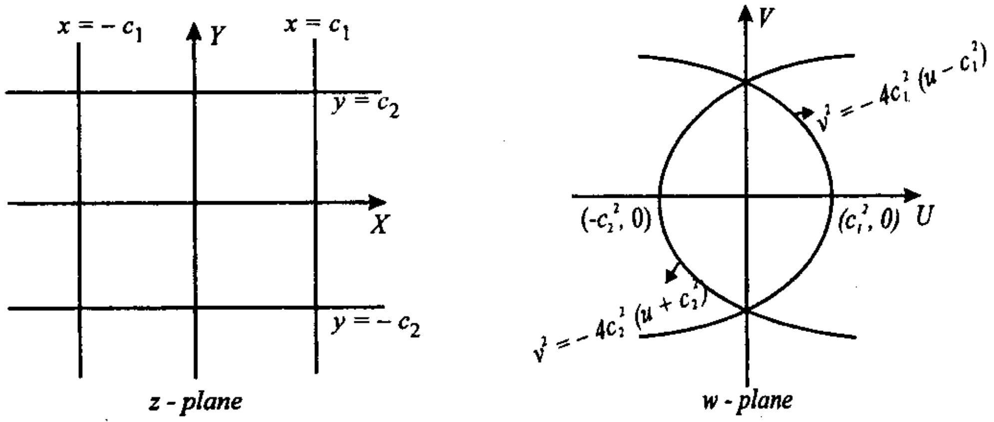
The set of equations (1) become $u = x^2 - c_2^2$, $v = 2xc_2$

Now $x = v/2c_2$ and substituting this in u we get $u = (v^2/4c_2^2) - c_2^2$

or $\quad v^2/4c_2^2 = u + c_2^2 \quad$ or $\quad v^2 = 4c_2^2(u + c_2^2)$

This is also a parabola in the w – plane symmetrical about the real axis whose vertex is at $(-c_2^2, 0)$ and focus at the origin. Also the line $y = -c_2$ is transformed into the same parabola.

Hence from these two cases we conclude that the straight lines parallel to the co-ordinate axes in the z – plane map onto parabolas in the w – plane.



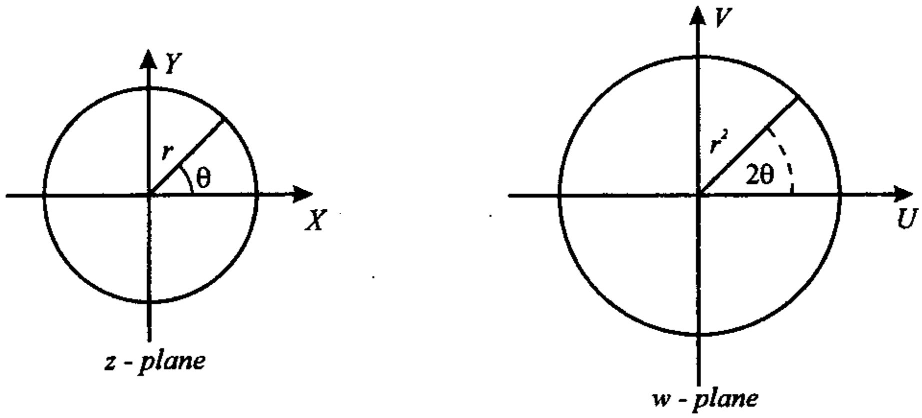
Case-3 : Let us consider a circle with centre origin and radius r in the z – plane.

ie., $\quad |z| = r \quad \therefore \quad z = re^{i\theta}$. Hence $w = z^2 = (re^{i\theta})^2$

ie., $\quad w = r^2 e^{2i\theta} = Re^{i\phi}$ *(say)* so that $R = r^2$ and $\phi = 2\theta$

This is also a circle in the w – plane having radius r^2 and subtending an angle 2θ at the origin.

Hence we conclude that a circle with centre origin and radius r in the z – plane maps onto a circle with centre origin and radius r^2 in the w – plane.



Case-4 : Let us consider a circle with centre a and radius r in the z – plane whose equation in the complex form is $|z-a| = r$

ie., $z-a = r e^{i\theta}$ or $z = a + r e^{i\theta}$

Hence $w = z^2 = (a + r e^{i\theta})^2 = a^2 + 2ar e^{i\theta} + r^2 e^{2i\theta}$

ie., $w - a^2 = 2ar e^{i\theta} + r^2 e^{2i\theta}$

Adding r^2 on both sides we have,

$$w - a^2 + r^2 = 2ar e^{i\theta} + r^2 (1 + e^{2i\theta})$$

ie., $w - (a^2 - r^2) = 2ar e^{i\theta} + r^2 (1 + e^{2i\theta})$

ie., $w - (a^2 - r^2) = 2r e^{i\theta} \left[a + \frac{r}{2} (e^{-i\theta} + e^{i\theta}) \right]$

ie., $w - (a^2 - r^2) = 2r e^{i\theta} (a + r\cos\theta)$

Suppose $w - (a^2 - r^2) = R e^{i\phi}$ then this equation becomes

$R e^{i\phi} = 2r e^{i\theta} (a + r\cos\theta)$ so that the pole in the w – plane is at the point $(a^2 - r^2)$.

Now, $R(\cos\phi + i\sin\phi) = 2r(a + r\cos\theta)(\cos\theta + i\sin\theta)$

$\therefore$ $R\cos\phi = 2r(a + r\cos\theta)\cos\theta$

$R\sin\phi = 2r(a + r\cos\theta)\sin\theta$

Squaring and adding these we have,

$$R^2(\cos^2\phi + \sin^2\phi) = [2r(a + r\cos\theta)]^2 (\cos^2\theta + \sin^2\theta)$$

ie., $R^2 = [2r(a + r\cos\theta)]^2$

$\Rightarrow$ $R = 2r(a + r\cos\theta)$ ...(1)

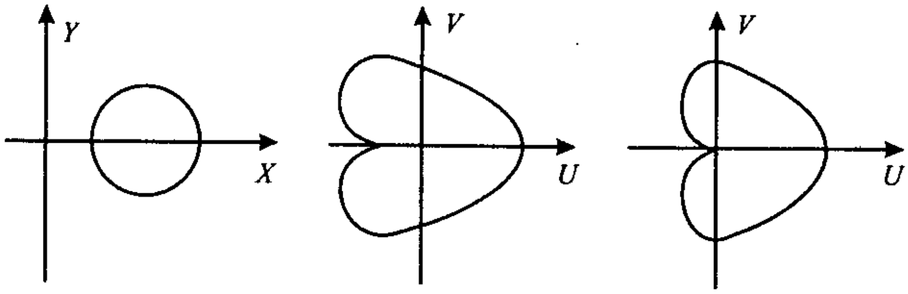
Also $R\sin\phi / R\cos\phi = \tan\theta$ or $\tan\phi = \tan\theta \Rightarrow \theta = \phi$ ...(2)

Using (2) in (1) we have $R = 2r(a + r\cos\phi)$ ...(3)

The curve given by (3) (*where* $a > r > 0$) is called a *Limacon. (Standard form being* $r = a + b\cos\theta$)

Hence, we conclude that a circle with centre *'a'* and radius r in the z – plane is mapped onto a Limacon in the w – plane.

In particular if $r = a$, (3) becomes $R = 2a^2(1 + \cos\phi)$ which is a *'cardioide'* in the w – plane [*Standard form being* $r = a(1 + \cos\theta)$]



z - plane w - plane (Limacon) w - plane (Cardioide)

Question-1 *Show that the transformation $w = z^2$ transforms the circle $|z-a| = r$ onto a limacon or cardioide.*

>> Case (4) as discussed already.

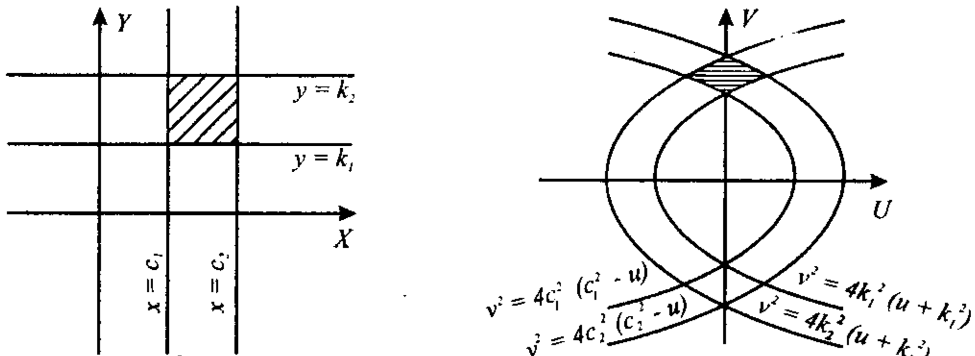
Question-2 *Find the images in the w− plane corresponding to the straight* lines $x = c_1$, $x = c_2$; $y = k_1$, $y = k_2$, *under the transformation $w = z^2$. Indicate the region with sketches.*

>> Discussions as in case 1 and 2.

The parabolas corresponding to $x = c_1$, $x = c_2$, $y = k_1$, $y = k_2$ are respectively the pairs

$$v^2 = -4c_1^2(u - c_1^2) \;;\quad v^2 = -4c_2^2(u - c_2^2)$$

and $$v^2 = 4k_1^2(u + k_1^2) \;;\quad v^2 = 4k_2^2(u + k_2^2)$$



z - plane w = plane

Question-3 *Find the region in the w− plane bounded by the lines $x = 1$, $y = 1$, $x + y = 1$ under the transformation $w = z^2$. Indicate the region with sketches.*

>> $w = z^2$

That is $u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

$\therefore \quad u = x^2 - y^2$ and $v = 2xy$ ... (1)

Consider $x = 1$: (1) becomes $u = 1 - y^2$, $v = 2y$

Substituting $v/2 = y$ in u we have $u = 1 - (v^2/4)$

ie., $v^2 = 4(1 - u)$. This is a parabola in the w - plane with vertex $(1, 0)$ and symmetrical about the u - axis.

Consider $y = 1$: (1) becomes $u = x^2 - 1$, $v = 2x$

Substituting $v/2 = x$ in u, we have $u = (v^2/4) - 1$

ie., $v^2 = 4(1 + u)$. This is also a parabola in the w - plane with vertex $(-1, 0)$ and symmetrical about the u - axis.

Consider $x + y = 1$ or $y = 1 - x$: (1) becomes

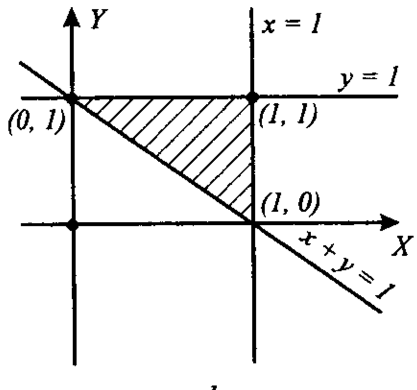
$$u = x^2 - (1 - x)^2 \text{ or } u = -1 + 2x \text{ and } v = 2x(1 - x)$$

Substituting $2x = 1 + u$ or $x = \frac{1}{2}(1 + u)$, v becomes

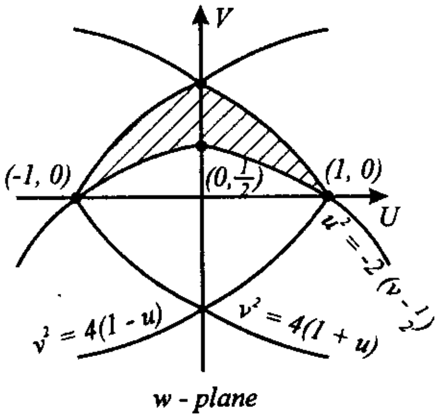
$$v = (1 + u)\left(1 - \frac{1 + u}{2}\right) = (1 + u)\cdot\frac{(1 - u)}{2} \text{ or } v = \frac{1}{2}(1 - u^2)$$

ie., $1 - u^2 = 2v$ or $u^2 = -2[v - (1/2)]$. This is also a parabola in the w - plane with vertex $(0, 1/2)$ symmetrical about the v - axis.

The region is as follows.



z- plane



w - plane

Question-4 *If $w = z^2$, sketch the family of the curves u = constant, v = constant. Show that the two families of curves intersect orthogonally.*

>> $w = z^2$ ie., $u + iv = (x+iy)^2 = (x^2 - y^2) + i(2xy)$

$\therefore \quad u = x^2 - y^2, \; v = 2xy$ $\quad \ldots (1)$

Let $\quad u = c_1, \; v = c_2$ (c_1 *and* c_2 *are constants*)

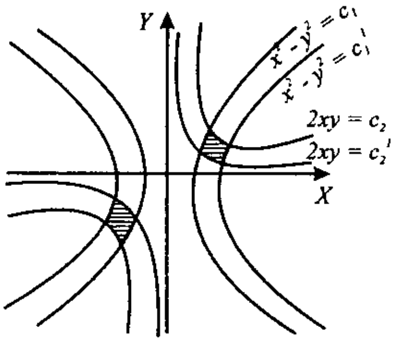
These represent lines parallel to the co-ordinate axes in the w – plane.

Hence (1) becomes $x^2 - y^2 = c_1, \; 2xy = c_2$

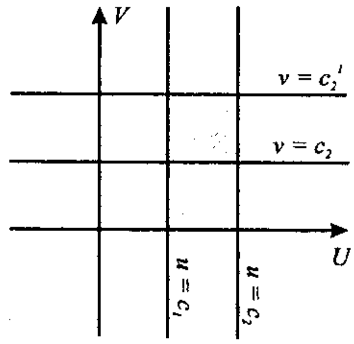
These are rectangular hyperbolas in the w -plane.

Also if $u = c_1', \; v = c_2'$ we have $x^2 - y^2 = c_1', \; 2xy = c_2'$.

The region bounded by these curves is as follows.



z- plane



w - plane

Now we shall show that $x^2 - y^2 = c_1$ and $2xy = c_2$ intersect orthogonally.

We need to show that the product of the slope of the tangents at the point of intersection is equal to -1.

$$x^2 - y^2 = c_1 \qquad ; \qquad 2xy = c_2$$

Differentiating these w.r.t. x we have,

$$2x - 2y\frac{dy}{dx} = 0 \qquad ; \qquad 2\left[x\frac{dy}{dx} + y\right] = 0$$

$$\therefore \quad \frac{dy}{dx} = m_1 = \frac{x}{y} \qquad ; \qquad \frac{dy}{dx} = m_2 = \frac{-y}{x}$$

Now $m_1 m_2 = \frac{x}{y} \cdot \frac{-y}{x} = -1.$

Hence the curves intersect orthogonally.

4.32 Discussion of $w = e^z$

Consider $w = e^z$

ie., $u + iv = e^{x+iy} = e^x \cdot e^{iy}$ or $u + iv = e^x(\cos y + i \sin y)$

$\therefore \quad u = e^x \cos y, \ v = e^x \sin y \qquad \ldots (1)$

We shall find the image in the w – plane corresponding to the straight lines parallel to the co-ordinate axes in the z – plane. That is x = constant and y = constant.

Let us eliminate x and y separately from (1).

Squaring and adding we get

$$u^2 + v^2 = e^{2x} \qquad \ldots (2)$$

Also by dividing we get $\dfrac{v}{u} = \dfrac{e^x \sin y}{e^x \cos y}$

ie., $\dfrac{v}{u} = \tan y \qquad \ldots (3)$

Case-1 : Let $x = c_1$ where c_1 is a constant.

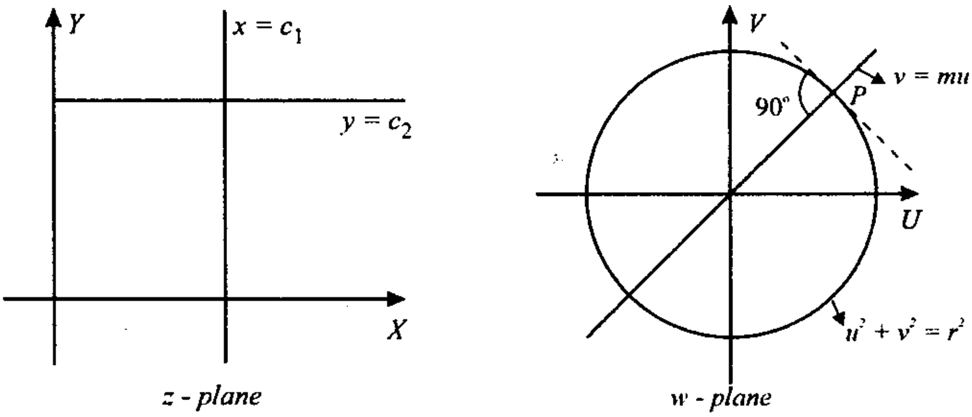
Equation (2) becomes $u^2 + v^2 = e^{2c_1}$ = constant $= r^2$ (*say*).

This represents a circle with centre origin and radius r in the w – plane.

Case-2 : Let $y = c_2$ where c_2 is a constant.

Equation (3) becomes $\dfrac{v}{u} = \tan c_2 = m\,(say) \quad \therefore \quad v = m\,u$

This represents a straight line passing through the origin in the w – plane.



Conclusion : The straight line parallel to the x – axis ($y = c_2$) in the z – plane maps onto a straight line passing through the origin in the w –plane. The straight line parallel to the y – axis ($x = c_1$) in the z – plane maps onto a circle with centre origin and radius r where $r = e^{c_1}$ in the w – plane.

Suppose we draw a tangent at the point of intersection of these two curves in the w – plane *(At P as in the figure) the angle subtended is equal to* 90°. *Hence these two curves can be regarded as orthogonal trajectories of each other.*

Question-1 *Show that the transformation* $w = e^z$ *map straight lines parallel to the co-ordinate axes in the* z *– plane onto orthogonal trajectories in the* w *– plane and sketch the region.*

>> Discussion of $w = e^z$ as done already.

Question-2 *Discuss the transformation* $w = e^z$ *with respect to the lines represented as co-ordinate axes in the* z *– plane.*

>> The co-ordinate axes in the z – plane are represented by $x = 0$, $y = 0$.

We have obtained $u^2 + v^2 = e^{2x}$ ...(2)

$$\frac{v}{u} = \tan y \qquad \ldots (3)$$

When $y = 0$ (3) becomes $v/u = \tan 0 = 0$ or $v = 0$

$\therefore$ the x – axis *(real axis)* in the z – plane is mapped onto the u – axis *(real axis)* in the w – plane.

When $x = 0$, (2) becomes $u^2 + v^2 = 1$

$\therefore$ the y axis in the z plane is mapped onto a unit circle with centre origin in the w – plane.

4.33 Discussion of $w = z + (a^2/z)$ and $w = z + (1/z)$, $z \neq 0$

>> Consider $w = z + (a^2/z)$.

Putting $z = r e^{i\theta}$ we have

$$u + iv = r e^{i\theta} + (a^2/r) e^{-i\theta}$$

ie., $$u + iv = r(\cos\theta + i\sin\theta) + (a^2/r)(\cos\theta - i\sin\theta)$$

ie., $$u + iv = [r + (a^2/r)]\cos\theta + i[r - (a^2/r)]\sin\theta$$

$$\Rightarrow \quad u = [r + (a^2/r)]\cos\theta \text{ and } v = [r - (a^2/r)]\sin\theta \qquad \ldots (1)$$

We shall eliminate r and θ separately from (1).

To eliminate θ let us put (1) in the form

$$\frac{u}{[r + (a^2/r)]} = \cos\theta \; ; \; \frac{v}{[r - (a^2/r)]} = \sin\theta$$

Squaring and adding we obtain

$$\frac{u^2}{[r+(a^2/r)]^2} + \frac{v^2}{[r-(a^2/r)]^2} = 1 \, , \, r \neq a \qquad \ldots (2)$$

To eliminate r let us put (1) in the form,

$$\frac{u}{\cos\theta} = [r+(a^2/r)] \; ; \; \frac{v}{\sin\theta} = [r-(a^2/r)]$$

Squaring and subtracting we obtain,

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = [r+(a^2/r)]^2 - [r-(a^2/r)]^2 = 4a^2$$

or $$\frac{u^2}{(2a\cos\theta)^2} - \frac{v^2}{(2a\sin\theta)^2} = 1 \qquad \ldots (3)$$

Since $z = re^{i\theta}$, $|z| = r$ and amp $z = \theta$

$$|z| = r \Rightarrow \sqrt{x^2+y^2} = r \text{ or } x^2+y^2 = r^2.$$

This represents a circle with centre origin and radius r in the z – plane when r is a constant.

$$\text{amp } z = \theta \Rightarrow \tan^{-1}(y/x) = \theta \quad \text{or} \quad y/x = \tan\theta.$$

This represents a straight line in the z – plane when θ is a constant.

We shall discuss the image in the w – plane, corresponding to $r =$ const. (*circle*) and $\theta =$ const. (*straight line*) in the z – plane.

Case - (1) Let $r =$ constant.

Equation (2) is of the form

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 \text{ where } A = [r+(a^2/r)], \; B = [r-(a^2/r)]$$

This represents an ellipse in the w – plane with foci $(\pm\sqrt{A^2-B^2}, 0) = (\pm 2a, 0)$

since $\sqrt{A^2-B^2} = \sqrt{[r+(a^2/r)]^2-[r-(a^2/r)]^2} = \sqrt{4a^2} = \pm 2a$

Hence we conclude that the circle $|z| = r =$ constant in the z – plane maps onto an ellipse in the w – plane with foci $(\pm 2a, 0)$.

Case - (2) Let $\theta =$ constant.

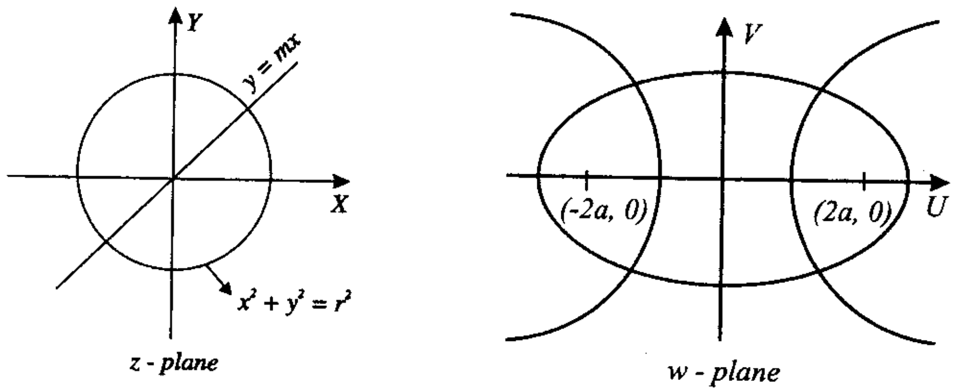
Equation (3) is of the form

$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1 \text{ where } A = 2a\cos\theta, \; B = 2a\sin\theta.$$

This represents a hyperbola in the w – plane with foci $(\pm\sqrt{A^2+B^2}, 0) = (\pm 2a, 0)$.

Hence we conclude that the straight line passing through the origin in the z – plane maps onto a hyperbola in the w – plane with foci $(\pm 2a, 0)$.

Since both these conics (*ellipse and hyperbola*) have the same foci independent of r, θ they are called confocal conics.



Question-1 *What are the points at which the transformation $w = z + (a^2/z)$ is not conformal ? Also show that this transformation maps the circle $|z| =$ constant and the straight line ampz = constant into confocal conics.*

>> $w = z + (a^2/z)$ $\quad \therefore \quad \dfrac{dw}{dz} = 1 - (a^2/z^2)$ or $\dfrac{dw}{dz} = \dfrac{z^2 - a^2}{z^2}$

$\dfrac{dw}{dz}$ will be equal to zero when $z^2 - a^2 = 0$ or $z = \pm a$.

Since $f'(z) = \dfrac{dw}{dz} \neq 0$ is the sufficient condition for the transformation $w = f(z)$ to be conformal the transformation $w = z + (a^2/z)$ is not conformal at the points $z = \pm a$.

Discussion of cases 1 and 2 make up the proof for the second part of the question.

Question-2 *Discuss the transformation $w = z + (1/z)$ with respect to the curves $r =$ constant $(\neq 0)$ and $\theta =$ constant $(\neq 0)$. Hence find the image of $r = 1$ and $\theta = \pi$ under this transformation.*

>> The given transformation $w = z + (1/z)$ is a particular case of the transformation $w = z + (a^2/z)$ where $a = 1$ discussed earlier.

Proceeding on the same lines we can say that the transformation $w = z + (1/z)$ map circles $|z| = r =$ constant and straight lines arg $z = \theta =$ constant in the z – plane into confocal conics in the w – plane with foci $(\pm 2, 0)$.

Now we shall discuss the case when $r = 1$ and $\theta = \pi$.

The transformation $w = z + (1/z)$ will give us,

$$u = [r + (1/r)] \cos\theta \text{ and } v = [r - (1/r)] \sin\theta$$

When $r = 1$, $u = 2\cos\theta$ and $v = 0$. Also if $\theta = \pi$ we have

$$u = -2, \; v = 0, \text{ since } \cos\pi = -1.$$

$v = 0$ represents the real axis ($u - axis$) in the w - plane.

But $|u| = 2|\cos\theta| \leq 2$ and hence the image consists of the segment of the real axis from -2 to 2 or $-2 \leq u \leq 2$.

Question-3 *Find the image of the circles* $|z| = 1$ and $|z| = 2$ [*Equivalently* $x^2 + y^2 = 1$, $x^2 + y^2 = 4$] *under the mapping* $w = z + (1/z)$

\>> $w = z + (1/z)$ will give us,

$$u = [r + (1/r)] \cos\theta \text{ and } v = [r - (1/r)] \sin\theta \quad \ldots (1)$$

Eliminating r, θ separately we obtain

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = [r + (1/r)]^2 - [r - (1/r)]^2 = 4$$

or

$$\frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1 \quad \ldots (2)$$

Also

$$\frac{u^2}{[r + (1/r)]^2} + \frac{v^2}{[r - (1/r)]^2} = 1, \; r \neq 1 \quad \ldots (3)$$

Consider $|z| = 1$ or $r = 1$.

Since we cannot use (3) we have from (1) $u = 2\cos\theta$, $v = 0$.

In the w - plane $v = 0$ represents the u - axis and we have,

$|u| = 2|\cos\theta| \leq 2$ or $-2 \leq u \leq 2$.

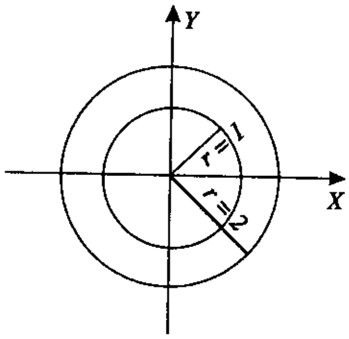
Hence we conclude that the circle $|z| = 1$ maps onto the segment of the real axis from -2 to 2 in the w - plane.

Next consider $|z| = 2$ or $r = 2$.

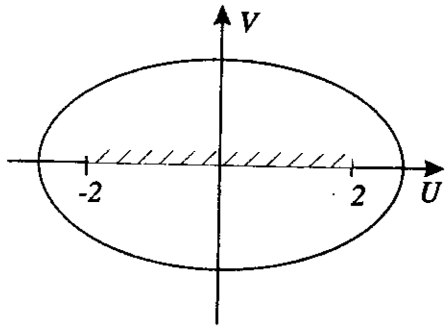
Substituting the value of r in (3) we get,

$$\frac{u^2}{(5/2)^2} + \frac{v^2}{(3/2)^2} = 1.$$

This is an ellipse in the w -plane. Hence we conclude that the circle $|z| = 2$ maps onto an ellipse in the w -plane.



z - plane



w - plane

EXERCISES

Find the bilinear transformation which map the points as given.

1. $z = 2, i, -2$ to $w = 1, i, -1$
2. $z = 1, i, -1$ to $w = i, 0, -1$
3. $z = 0, -i, -1$ to $w = i, 1, 0$
4. $z = 2, 1, 0$ to $w = 1, 0, i$
5. $z = \infty, i, 0$ to $w = 0, i, \infty$, find also the invariant points.
6. $z = 1, i, -1$ to $w = 0, i, \infty$

Find the invariant points of the following bilinear transformations [7 to 10]

7. $w = \dfrac{z-1-i}{z+2}$

8. $w = \dfrac{3z - 5i}{iz - 1}$

9. $w = \dfrac{3z - 4}{z-1}$

10. $w = \dfrac{3iz + 1}{z+i}$

11. Show that there are two points which are left invariant by the general bilinear transformation. What is the condition that
    (i) these two points coincide ?
    (ii) these are two finite fixed points
    (iii) one finite and another infinite fixed point
    (iv) only one infinite fixed point.

12. Prove that $w = z/1-z$ maps the upper half of the z-plane onto the upperhalf of the w-plane.

    [Hint : *Express* z *in terms of* w *and find* x, y *to discuss for* $y \geq 0$]

13. Show that the transformation $w = z - i/1 - iz$ maps the unit circle with centre origin in the z-plane onto the real axis in the w-plane.

[Hint : *Express* z *in terms of* w and *consider* $x^2 + y^2 = 1$ *written in the form* $z\bar{z} = 1$]

14. Prove that $w = 1 + z/1 - z$ maps the region $|z| \le 1$ onto the half plane $R(u) \ge 0$ being the region $u \ge 0$.

15. Given $w = z - i/iz - 1$, show that the unit circle with centre origin in the w-plane is mapped onto the imaginary axis in the z-plane.

16. Obtain the image of the region bounded by the lines $x = 1,\ x = 2,\ y = 1,\ y = 2$ under the transformation $w = e^z$ & sketch the region.

17. If $w = x + i(by/a),\ 0 < a < b$, prove that the inside of the circle $x^2 + y^2 = a^2$ corresponds to the inside of an ellipse in the w-plane.

18. Given $w = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ show that,

    (a) the transformation is not conformal at $z = \pm 1$.

    (b) the transformation maps the circle $|z| =$ constant and the straight line $\arg z =$ constant into confocal conics with foci $(\pm 1, 0)$.

ANSWERS

1. $w = \dfrac{3z + 2i}{iz + 6}$

2. $w = \dfrac{iz + 1}{(1 - i)z}$

3. $w = i\left(\dfrac{1 + z}{1 - z}\right)$

4. $w = \dfrac{2(z - 1)}{-iz + (1 + 2i)}$

5. $w = \dfrac{-1}{z}$; $\pm i$

6. $w = \dfrac{z - 1}{z + 1}$

7. $-i, i - 1$

8. $i, -5i$

9. 2

10. i

11. Invariant points are $z = \dfrac{(d - a) \pm \sqrt{(d - a)^2 + 4bc}}{2c}$ where $w = \dfrac{az + b}{cz + d}$

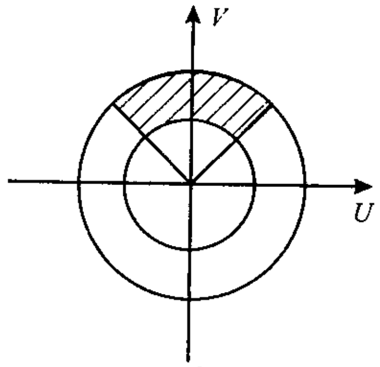
    (i) $c \ne 0,\ (d - a)^2 + 4bc = 0$

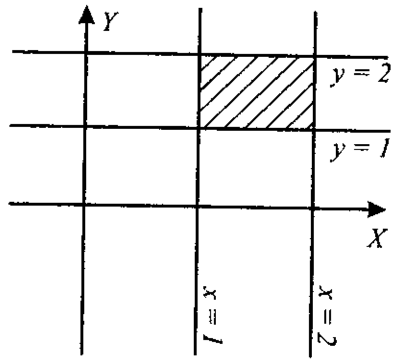
    (ii) $c \ne 0,\ (d - a)^2 + 4bc \ne 0$

    (iii) $c = 0,\ d - a \ne 0$

    (iv) $c = 0,\ d - a = 0$

16. $u^2 + v^2 = e^2;\ u^2 + v^2 = e^4\ ;\ v/u = \tan 1,\ v/u = \tan 2.$



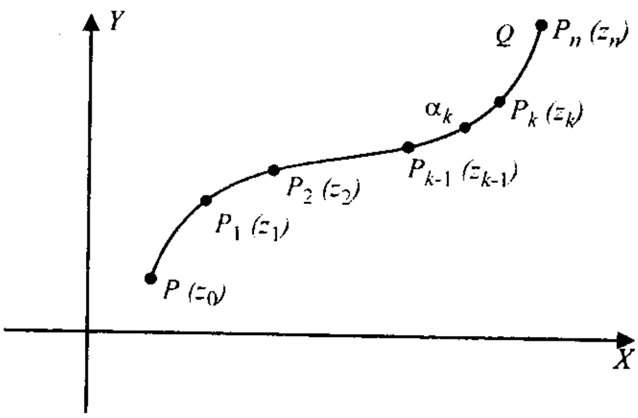


4.4 Complex Integration

4.41 Introduction

We have already studied the topic *Vector Integration* in which we are acquainted with various concepts and theorems associated with the vector line integral defined over a curve C. In this topic we study the integration of complex valued functions defined along curves in the complex plane.

4.42 Complex line integral



Consider a continuous function $f(z)$ of the complex variable $z = x + iy$ defined at all points of a curve C extending from P to Q. Divide the curve C into n parts by arbitrarily taking points $P = P(z_0),\ P_1(z_1),\ P_2(z_2)\ \ldots\ P_k(z_k)\ \ldots,$ $P_n(z_n) = Q$ on the curve C. Let α_k be any point on the arc of the curve from P_{k-1} to P_k and let $\delta z_k = z_k - z_{k-1}$ where $k = 1, 2, 3, \ldots n$.

Then $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} f(\alpha_k)\, \delta z_k$ where $\max |\delta z_k| \to 0$ as $n \to \infty$ is defined as the ***complex line integral*** along the path C usually denoted by $\int_C f(z)\, dz$.

If C is a simple closed curve the notation $\oint_C f(z)\, dz$ is also used.

4.43 Properties of complex integral

(i) If $-C$ denotes the curve traversed from Q to P then

$$\int_{-C} f(z)\, dz = -\int_C f(z)\, dz$$

(ii) If C is split into a number of parts $C_1, C_2, C_3, \ldots$, then

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \int_{C_3} f(z)\, dz + \cdots$$

(iii) If λ_1 and λ_2 are constants then

$$\int_C [\lambda_1 f_1(z) \pm \lambda_2 f_2(z)]\, dz = \lambda_1 \int_C f_1(z)\, dz \pm \lambda_2 \int_C f_2(z)\, dz$$

4.44 Line integral of a complex valued function

Let $f(z) = u(x, y) + i\, v(x, y)$ be a complex valued function defined over a region R and C be a curve in the region. Then

$$\int_C f(z)\, dz = \int_C (u + iv)\,(dx + i\,dy)$$

i.e.,

$$\int_C f(z)\, dz = \int_C (u\,dx - v\,dy) + i \int_C (v\,dx + u\,dy)$$

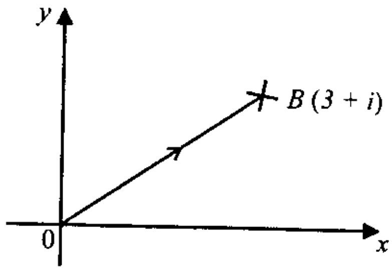
This shows that the evaluation of a line integral of a complex valued function is nothing but the evaluation of line integrals of real valued functions.

WORKED PROBLEMS

14. *Evaluate* $\int_C z^2\, dz$

(a) *along the straight line from* $z = 0$ *to* $z = 3 + i$

(b) *along the curve made up of two line segments, one from* $z = 0$ to $z = 3$ *and another from* $z = 3$ *to* $z = 3 + i$.



>> **(a)** $\int_C z^2\,dz = \int_{z=0}^{3+i} z^2\,dz$

Here z varies from 0 to $3+i$ means that (x, y) varies from $(0, 0)$ to $(3, 1)$. The equation of the line joining $(0, 0)$ and $(3, 1)$ is given by

$$\frac{y-0}{x-0} = \frac{1-0}{3-0} \quad \text{or} \quad y = \frac{x}{3}$$

Further $z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$ and $dz = dx + idy$

$$\int_C z^2\,dz = \int_{(0,0)}^{(3,1)} \{(x^2 - y^2) + i(2xy)\}(dx + i\,dy)$$

$$= \int_{(0,0)}^{(3,1)} \{(x^2 - y^2)\,dx - 2xy\,dy\} + i\int_{(0,0)}^{(3,1)} \{2xy\,dx + (x^2 - y^2)\,dy\}$$

We have $y = \dfrac{x}{3}$ or $x = 3y$ and we shall convert these integrals into the variable y and integrate w.r.t. y from 0 to 1. We also have $dx = 3dy$

$$\therefore \quad \int_C z^2\,dz = \int_{y=0}^{1} \{(9y^2 - y^2)\,3dy - 2(3y)y\,dy\}$$

$$+\, i\int_{y=0}^{1} \{2(3y)\,y\cdot 3dy + (9y^2 - y^2)\,dy\}$$

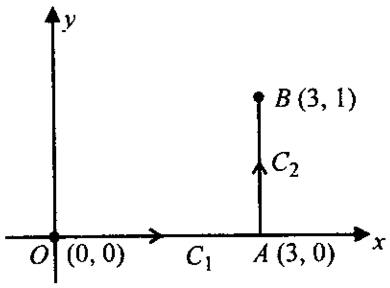
$$= \int_{y=0}^{1} (24y^2 - 6y^2)\,dy + i\int_{y=0}^{1} (18y^2 + 8y^2)\,dy$$

$$= \int_0^1 18y^2\,dy + i\int_0^1 26y^2\,dy$$

$$= 18\left[\frac{y^3}{3}\right]_0^1 + 26i\left[\frac{y^3}{3}\right]_0^1 = 6 + \frac{26}{3}i$$

Thus $\int_C z^2\,dz = 6 + \dfrac{26}{3}i$ **along the given path.**

(b) Segments from $z = 0$ to $z = 3$ and then from $z = 3$ to $3 + i$ means that (x, y) varies from $(0, 0)$ to $(3, 0)$ and then from $(3, 0)$ to $(3, 1)$ as shown in the following figure.



$$\int_C z^2\,dz = \int_{C_1} z^2\,dz + \int_{C_2} z^2\,dz \qquad \ldots (1)$$

Now along C_1 : $y = 0 \Rightarrow dy = 0$ and x varies from 0 to 3. $z^2\,dz$ becomes $x^2\,dx$

Also along C_2 : $x = 3 \Rightarrow dx = 0$ and y varies from 0 to 1.

$z^2\,dz$ becomes $(3 + iy)^2\,i\,dy$. Now (1) becomes,

$$\int_C z^2\,dz = \int_{x=0}^{3} x^2\,dx + i\int_{y=0}^{1} (3+iy)^2\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^3 + i\int_{y=0}^{1} (9 - y^2 + 6iy)\,dy$$

$$= 9 + i\left[9y - \frac{y^3}{3} + 3i\,y^2\right]_0^1$$

$$= 9 + i\left(9 - \frac{1}{3} + 3i\right) = (9 - 3) + i\cdot\frac{26}{3}$$

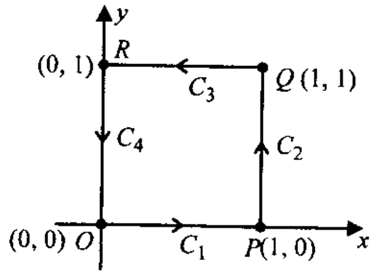
Thus $\displaystyle\int_C z^2\,dz = 6 + \frac{26}{3}\,i$ along the given path.

- -

15. *Evaluate* $\int_C |z|^2 dz$ *where* C *is a square with the following vertices.*

$(0, 0)\ (1, 0)\ (1, 1)\ (0, 1)$.

>> The curve C is as shown in the following figure.



$$\int_C |z|^2 dz = \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz \quad \dots(1)$$

We have $|z|^2 dz = (x^2 + y^2)(dx + i\,dy)$

Along $OP\ (C_1)$, $y = 0 \Rightarrow dy = 0$. $|z|^2 dz = x^2 dx$ where $0 \le x \le 1$

Along $PQ\ (C_2)$, $x = 1 \Rightarrow dx = 0$. $|z|^2 dz = (1 + y^2)\, i\, dy$ where $0 \le y \le 1$

Along $QR\ (C_3)$, $y = 1 \Rightarrow dy = 0$. $|z|^2 dz = (x^2 + 1)\, dx$ where $1 \le x \le 0$

Along $RO\ (C_4)$, $x = 0 \Rightarrow dx = 0$. $|z|^2 dz = y^2 (i\, dy)$ where $1 \le y \le 0$

Using these results in (1) we obtain

$$\int_C |z|^2 dz = \int_{x=0}^{1} x^2 dx + i \int_{y=0}^{1} (1 + y^2)\, dy + \int_{x=1}^{0} (x^2 + 1)\, dx + i \int_{y=1}^{0} y^2\, dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + i\left[y + \frac{y^3}{3}\right]_0^1 + \left[\frac{x^3}{3} + x\right]_1^0 + i\left[\frac{y^3}{3}\right]_1^0$$

$$= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = -1 + i$$

Thus $\int_C |z|^2 dz = -1 + i$ along the given path.

16. *Evaluate* $\int_0^{2+i} (\bar{z})^2 \, dz$ *along :*

(a) *the line* $x = 2y$, **(b)** *the real axis upto 2 and then vertically to* $2+i$

>> Let $I = \int_0^{2+i} (\bar{z})^2 dz$

We have $(\bar{z})^2 = (x - iy)^2 = (x^2 - y^2) - i(2xy)$ ...(1)

and $dz = dx + i\,dy$ ...(2)

(a) Along $x = 2y$, $dx = 2\,dy$

$z = 0$ to $2+i \Rightarrow (x, y)$ varies from $(0, 0)$ to $(2, 1)$ where $0 \le y \le 1$

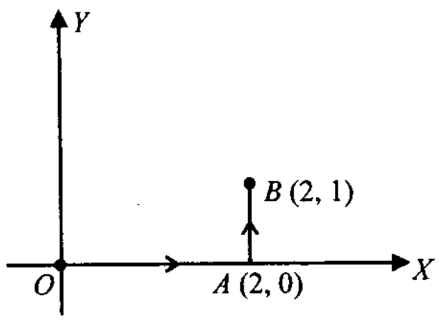
$$\therefore \quad I = \int_{y=0}^{1} \left[(4y^2 - y^2) - i.4y^2 \right] (2\,dy + i\,dy)$$

$$= \int_{y=0}^{1} (3 - 4i) y^2 (2 + i) \, dy$$

$$= \int_{y=0}^{1} (10 - 5i) y^2 \, dy = 5(2 - i) \left[\frac{y^3}{3} \right]_0^1 = \frac{5}{3}(2 - i)$$

Thus $I = \frac{5}{3}(2 - i)$ along the given path.

(b) $I = \int_{OA} (\bar{z})^2 dz + \int_{AB} (\bar{z})^2 \, dz$ ...(3)



Along OA where $O = (0, 0)$ & $A = (2, 0)$; $y = 0 \Rightarrow dy = 0$ & $0 \le x \le 2$

Along AB where $A = (2, 0)$ & $B = (2, 1)$; $x = 2 \Rightarrow dx = 0$ and $0 \le y \le 1$

From (1) and (2) we have,

along OA, $(\bar{z})^2 dz = x^2 dx$; $0 \le x \le 2$

along AB, $(\bar{z})^2 dz = \left[(4-y^2)-4iy\right] i\, dy$; $0 \le y \le 1$

$$\int_{OA} (\bar{z})^2 dz = \int_{x=0}^{2} x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} \qquad \ldots (4)$$

$$\int_{AB} (\bar{z})^2 dz = i \int_{y=0}^{1} [(4-y^2)-4iy]\, dy = i\left[4y - \frac{y^3}{3}\right]_0^1 + 4\left[\frac{y^2}{2}\right]_0^1$$

$$\therefore \quad \int_{AB} (\bar{z})^2 dz = 2 + \frac{11}{3} i \qquad \ldots (5)$$

Using (4) and (5) in (3) we have, $I = \frac{8}{3} + \left(2 + \frac{11}{3} i\right)$

Thus $\boldsymbol{I = \frac{1}{3}(14 + 11i)}$ along the given path.

---

17. *Evaluate* $\int_{(0,3)}^{(2,4)} (2y+x^2)\, dx + (3x-y)\, dy$ *along the following paths.*

   (a) *the parabola* $x = 2t$, $y = t^2 + 3$
   (b) *the straight line from* $(0, 3)$ *to* $(2, 4)$

>> **(a)** x varies from 0 to 2 and hence

$$\left.\begin{array}{l} \text{if } x = 0,\ 2t = 0 \quad \therefore\ t = 0 \\ \text{if } x = 2,\ 2t = 2 \quad \therefore\ t = 1 \end{array}\right\} \Rightarrow t \text{ varies from 0 to 1.}$$

$$I = \int_{(0,3)}^{(2,4)} (2y+x^2)\, dx + (3x-y)\, dy$$

$$\therefore \quad I = \int_{t=0}^{1} \left\{2(t^2+3) + 4t^2\right\} 2dt + \left\{3(2t) - (t^2+3)\right\} 2t\, dt$$

$$I = \int_0^1 \left[2(6t^2 + 6) + (6t - t^2 - 3)\,2t \right] dt$$

$$= \int_0^1 (24t^2 - 2t^3 - 6t + 12)\, dt$$

$$= 24\left[\frac{t^3}{3}\right]_0^1 - 2\left[\frac{t^4}{4}\right]_0^1 - 6\left[\frac{t^2}{2}\right]_0^1 + 12\,[t]_0^1$$

$$= 8 - \frac{1}{2} - 3 + 12 = \frac{33}{2}$$

Thus $I = 33/2$ along the given path.

(b) Equation of the straight line joining (0, 3) and (2, 4) is given by

$$\frac{y-3}{x-0} = \frac{4-3}{2-0}$$

i.e., $\frac{y-3}{x} = \frac{1}{2}$ or $x = 2y - 6$. Hence $dx = 2\,dy$.

Now, $I = \int_{y=3}^{4} \{2y + (2y-6)^2\}\, 2\,dy + \{3(2y-6) - y\}\, dy$

$$= \int_3^4 \{(4y^2 - 22y + 36)\,2 + (5y - 18)\}\, dy$$

$$= \int_3^4 (8y^2 - 39y + 54)\, dy$$

$$= 8\left[\frac{y^3}{3}\right]_3^4 - 39\left[\frac{y^2}{2}\right]_3^4 + 54\,[y]_3^4$$

$$= \frac{8}{3}(64 - 27) - \frac{39}{2}(16 - 9) + 54\,(4 - 3)$$

$$= \frac{296}{3} - \frac{273}{2} + 54 = \frac{97}{6}$$

Thus $I = 97/6$ along the given path.

18. *Evaluate* $\int_C \bar{z}\, dz$ *where C represents the following paths.*

(*a*) *the straight line from* $-i$ *to* i

(*b*) *the right half of the unit circle* $|z| = 1$ *from* $-i$ *to* i

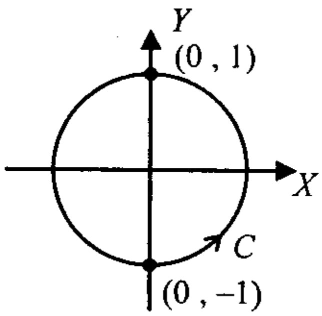
>> **(a)** $z = x + iy \quad \therefore \quad \bar{z} = x - iy,\ dz = dx + i\,dy$, C is the straight line joining the points $(0, -1)$ and $(0, 1)$ Here $x = 0 \Rightarrow dx = 0$, y varies from -1 to $+1$.

$$\int_C \bar{z}\, dz = \int_{y=-1}^{+1} (x - iy)(dx + i\,dy)$$

$$= \int_{y=-1}^{+1} (-iy)(i\,dy) = \int_{-1}^{+1} y\,dy = \left[\frac{y^2}{2}\right]_{-1}^{+1} = \frac{1}{2} - \frac{1}{2} = 0$$

Thus $\int_C \bar{z}\, dz = 0$ along the given path.

(b) The curve C is shown in the following figure.



$C : |z| = 1$. We can take $z = e^{i\theta}$

Also $\bar{z} = e^{-i\theta}$ and $dz = i e^{i\theta} d\theta$

From the figure y varies from -1 to $+1$ and $x = 0$

But $x = \cos\theta,\ y = \sin\theta$

$y = -1 \ :\ \sin\theta = -1 \quad \therefore \quad \theta = -\pi/2$

$y = +1 \ :\ \sin\theta = +1 \quad \therefore \quad \theta = \pi/2$

Now $$\int_C \bar{z}\, dz = \int_{\theta=-\pi/2}^{\pi/2} e^{-i\theta} \cdot i e^{i\theta}\, d\theta = i \int_{-\pi/2}^{\pi/2} 1 \cdot d\theta = i\,[\theta]_{-\pi/2}^{\pi/2} = \pi i$$

Thus $\int_C \bar{z}\, dz = \pi i$ along the given path.

---

19. *Evaluate* $\int_{1-i}^{2+i} (2x + iy + 1)\, dz$ *along the following paths :*

(*a*) $x = t + 1,\ y = 2t^2 - 1$

(*b*) *straight line joining* $(1 - i)$ *and* $(2 + i)$

>> **(a)** $x = t + 1,\ y = 2t^2 - 1$

$\therefore \quad dx = dt,\ dy = 4t\,dt.$ x varies from 1 to 2.

If $x=1,\quad t+1=1 \quad \therefore \quad t=0$

$x=2,\ t+1=2 \quad \therefore \quad t=1$

Also $dz = dx + i\,dy$

Let the given integral be denoted by I so that we have,

$$I = \int_{t=0}^{1} \{2(t+1) + i(2t^2-1) + 1\}\{dt + i\,4t\,dt\}$$

$$= \int_{t=0}^{1} (2t + 2it^2 + 3 - i)(1 + 4it)\,dt$$

$$= \int_{0}^{1} \{-8t^3 + 10\,it^2 + (12i+6)t + (3-i)\}\,dt$$

$$= -8\left[\frac{t^4}{4}\right]_0^1 + 10\,i\left[\frac{t^3}{3}\right]_0^1 + 6(2i+1)\left[\frac{t^2}{2}\right]_0^1 + (3-i)[t]_0^1$$

$$= -2 + \frac{10\,i}{3} + 3(2i+1) + (3-i) = 4 + \frac{25\,i}{3}$$

Thus $I = \mathbf{4} + \dfrac{\mathbf{25\,i}}{\mathbf{3}}$ along the given path.

(b) Equation of the straight line joining $(1,-1)$ and $(2,1)$ is given by

$$\frac{y+1}{x-1} = \frac{1-(-1)}{2-1}$$

i.e., $\dfrac{y+1}{x-1} = 2$ or $y+1 = 2x-2$ or $y = 2x-3$

Hence the equation of the straight line is $y = 2x - 3 \quad \therefore \quad dy = 2\,dx$

Now $I = \displaystyle\int_{x=1}^{2} \{2x + i(2x-3) + 1\}\{dx + i\cdot 2\,dx\}$

$$= \int_{x=1}^{2} \{2(1+i)x + (1-3i)\}(1+2i)\,dx$$

$$= \int_{x=1}^{2} \{2(1+i)(1+2i)x + (1-3i)(1+2i)\}\,dx$$

$$I = \int_{x=1}^{2} \{2(-1+3i)x + (1-3i)(1+2i)\}\, dx$$

$$= (1-3i) \int_{x=1}^{2} \{-2x + (1+2i)\}\, dx$$

$$= (1-3i) \left\{[-x^2]_1^2 + (1+2i)\,[x]_1^2\right\}$$

$$= (1-3i)\{-3+1+2i\} = (1-3i)\,2(i-1) = 2(2+4i)$$

Thus **$I = 4(1+2i)$** along the given path.

---

20. *If C is a cirlce with centre' a' and radius 'r' then show that*

(a) $\int_C \frac{dz}{z-a} = 2\pi i$ *(b)* $\int_C (z-a)^n\, dz = 0$ *if* $n \neq -1$

or

Show that $\int_C (z-a)^n\, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$ *where C is the circle* $|z-a| = r$.

>> On the given circle $|z-a| = r$, we have $z-a = re^{i\theta}$, Hence $dz = i\,r\,e^{i\theta}\, d\theta$
Also, $0 \le \theta \le 2\pi$

(a) $$\int_C \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{i\,r\,e^{i\theta}\, d\theta}{r\,e^{i\theta}} = i \int_{\theta=0}^{2\pi} d\theta = i\,[\theta]_0^{2\pi} = 2\pi i$$

Thus $\int_C \frac{dz}{z-a} = 2\pi i$

(b) Also $$\int_C (z-a)^n\, dz = \int_{\theta=0}^{2\pi} (r\,e^{i\theta})^n\, i\,r\,e^{i\theta}\, d\theta$$

$$= i\,r^{n+1} \int_{\theta=0}^{2\pi} e^{i(n+1)\theta}\, d\theta$$

$$= i\,r^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_{\theta=0}^{2\pi}$$

$$\int_C (z-a)^n\, dz = \frac{r^{n+1}}{n+1}\,[e^{i(n+1)2\pi} - 1]$$

But $e^{i(n+1)2\pi} = \cos(n+1)2\pi + i\sin(n+1)\,2\pi = 1 + i\cdot 0 = 1$

$\because$ $\cos 2k\pi = +1$ and $\sin 2k\pi = 0$ for $k = 1, 2, 3, \ldots$

Hence $\int_C (z-a)^n\, dz = \dfrac{r^{n+1}}{n+1}[1-1] = 0$ where $n \neq -1$

Thus we have proved that, $\displaystyle\int_C (z-a)^n\, dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$

4.5 Cauchy's theorem

Statement: If $f(z)$ ***is analytic at all points inside and on a simple closed curve*** C

then $\int_C f(z)\, dz = 0.$

Proof: Let $f(z) = u + iv$

Then $\int_C f(z)\, dz = \int_C (u+iv)(dx + i\,dy)$

i.e., $\int_C f(z)\, dz = \int_C (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$ ...(1)

We have Green's theorem in a plane stating that if $M(x,y)$ and $N(x,y)$ are two real valued functions having continuous first order partial derivatives in a region R bounded by the curve C then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Applying this theorem to the two line integrals in the RHS of (1) we obtain,

$$\int_C f(z)\, dz = \iint_R \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Since $f(z)$ is analytic, we have Cauchy-Riemann equations :

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial v}{\partial x} = \dfrac{-\partial u}{\partial y}$ and hence we have,

$$\int_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\, dy$$

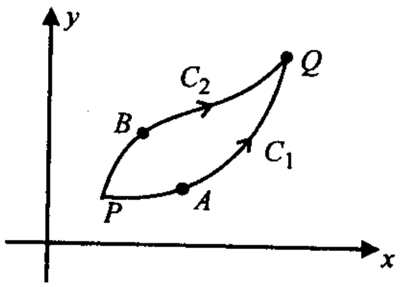
Thus we get $\int_C f(z)\, dz = 0.$

This proves Cauchy's theorem.

4.51 Consequences of Cauchy's theorem

1. **Statement** *If $f(z)$ is analytic in a region R and if P and Q are any two points in it then $\int_P^Q f(z)\,dz$ is independent of the path joining P and Q. That is $\int_P^Q f(z)\,dz$ is same for all curves joining P and Q.*

Proof : Let C_1 and C_2 be two simple curves joining P and Q such that both the curves lie in the region R. Then their union $PAQBP$ as in the following figure below becomes a simple closed curve C in the region R.



Now by Cauchy's theorem $\int_C f(z)\,dz = 0$

i.e., $\int_{PAQ} f(z)\,dz + \int_{QBP} f(z)\,dz = 0$

i.e., $\int_{PAQ} f(z)\,dz - \int_{PBQ} f(z)\,dz = 0$

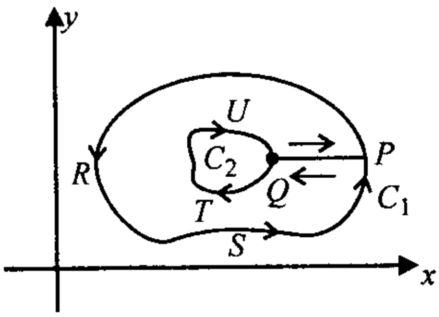
or $\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$

This implies that $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$

---

2. **Statement** : *If C_1, C_2 are two simple closed curves such that C_2 lies entirely within C_1 and if $f(z)$ is analytic on C_1, C_2 and in the region bounded by C_1, C_2 (known as the annular region) then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$*

Proof :



Let us introduce a cross-cut in the form of a line segment PQ with the point P on C_1 and Q on C_2. Then the curve $PRSPQTUQP$ as shown in the figure is a simple closed curve and $f(z)$ is analytic inside and on the boundary of C.

Hence by Cauchy's theorem $\int_C f(z)\, dz = 0.$

Since C is the union of the arcs $PRSP$, PQ, $QTUQ$ and QP, the theorem becomes

$$\int_{C_1} f(z)\, dz + \int_{PQ} f(z)\, dz + \int_{-C_2} f(z)\, dz + \int_{QP} f(z)\, dz = 0$$

i.e.,
$$\int_{C_1} f(z)\, dz + \int_{PQ} f(z)\, dz - \int_{C_2} f(z)\, dz - \int_{PQ} f(z)\, dz = 0$$

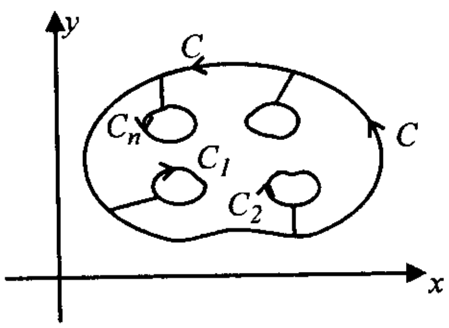
Thus $\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$

---

3. **Statement** : *If C is a simple closed curve enclosing non overlapping simple closed curves $C_1, C_2, C_3, \ldots C_n$ and if $f(z)$ is analytic in the annular region between C and these curves then*

$$\int_C f(z)\, dz = \int_{C_1} f(z) + \int_{C_2} f(z)\, dz + \cdots + \int_{C_n} f(z)\, dz$$

Proof :



Let us introduce cross cuts from C to each of the curves $C_1, C_2, \ldots C_n$. We then get a simple closed curve Γ made up of $C, C_1, C_2, \ldots C_n$ where $f(z)$ is analytic inside and on Γ

$\therefore$ by Cauchy's theorem $\int_{\Gamma} f(z)\, dz = 0.$

i.e.,
$$\int_C f(z)\, dz + \int_{-C_1} f(z)\, dz + \int_{-C_2} f(z)\, dz + \cdots + \int_{-C_n} f(z)\, dz = 0$$

Here the integrals along the cross cuts cancel with each other because of the direction being opposite.

$$\therefore \quad \int_C f(z)\, dz - \int_{C_1} f(z)\, dz - \int_{C_2} f(z)\, dz \cdots - \int_{C_n} f(z)\, dz = 0$$

Thus we have proved that

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \cdots + \int_{C_n} f(z)\, dz$$

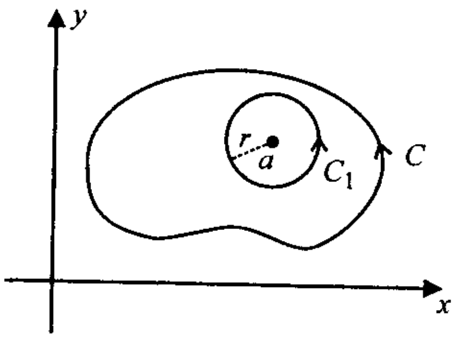
4.6 Cauchy's integral formula

If $f(z)$ is analytic inside and on a simple closed curve C and if 'a' is any point within C then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

Proof : Since 'a' is a point within C, we shall enclose it by a circle C_1 with $z = a$ as centre and r as radius such that C_1 lies entirely within C.

The function $\dfrac{f(z)}{z-a}$ is analytic inside and on the boundary of the annular region between C and C_1



Now, as a consequence of Cauchy's theorem,

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \qquad \ldots(1)$$

The equation of C_1 (*circle with centre 'a' and radius* r) can be written in the form $|z-a| = r$. That is equivalent to,

$$z - a = re^{i\theta} \text{ or } z = a + re^{i\theta},\ 0 \le \theta \le 2\pi\ \ dz = ire^{i\theta}\,d\theta.$$

Using these results in the RHS of (1) we have,

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

i.e., $$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

This is true for any $r > 0$ however small. Hence as $r \to 0$ we get,

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a)\,d\theta = if(a)\,[\theta]_0^{2\pi} = 2\pi i f(a)$$

Thus $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz \quad [\textit{Cauchy's integral formula}]$$

4.61 Generalized Cauchy's integral formula

If $f(z)$ is analytic inside and on a simple closed curve C and if 'a' is a point within C then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Proof : We have Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\,dz \qquad \ldots(1)$$

Applying Leibnitz rule for differentiation under the integral sign we have,

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\,\frac{\partial}{\partial a}\left[\frac{1}{z-a}\right]dz$$

i.e., $f'(a) = \frac{1}{2\pi i} \int_C f(z) \cdot \left\{(-1)(z-a)^{-2} \cdot (-1)\right\} dz$

i.e., $f'(a) = \frac{1!}{2\pi i} \int_C \frac{f(z)}{(z-a)^2}$ ...(2)

Applying Leibnitz rule once again for (2) we obtain

$$f''(a) = \frac{1!}{2\pi i} \int_C f(z) \frac{\partial}{\partial a} [(z-a)^{-2}] \, dz$$

$$= \frac{1!}{2\pi i} \int_C f(z) \cdot (-2)(z-a)^{-3} (-1) \, dz$$

i.e., $f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz$ ...(3)

Continuing like this, after differentiating n times we obtain,

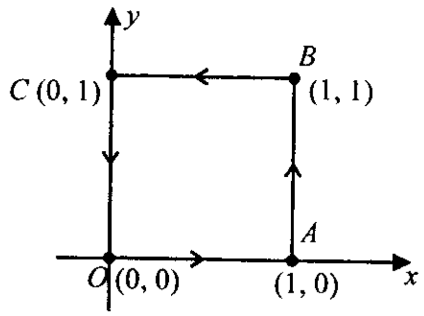
$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

Here $f^{(n)}(a)$ denotes the n^{th} derivative of $f(z)$ at $z = a$.

WORKED PROBLEMS

21. *Verify Cauchy's theorem for the function $f(z) = z^2$ where C is the square having vertices $(0, 0)(1, 0)(1, 1)(0, 1)$.*

>>



C is the square $OABC$ and we have by Cauchy's theorem $\int_C f(z)\, dz = 0$.

Therefore we have to show that,

$$\int_{OA} z^2 dz + \int_{AB} z^2 dz + \int_{BC} z^2 dz + \int_{CO} z^2 dz = 0$$

Along OA, $y = 0 \quad \therefore \quad dy = 0 \ ; \ 0 \le x \le 1$

$$z^2\,dz = (x + iy)^2\,(dx + i\,dy) = x^2\,dx$$

$$\therefore \qquad \int_{OA} z^2\,dz = \int_{x=0}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\int_{OA} z^2\,dz = \frac{1}{3} \qquad \ldots (1)$$

Along AB, $x = 1$, $\therefore \quad dx = 0 \ ; \ 0 \le y \le 1$

$$z^2\,dz = (x + iy)^2\,(dx + i\,dy) = (1 + iy)^2\,i\,dy$$

$$\int_{AB} z^2\,dz = i \int_{y=0}^{1} (1 + iy)^2\,dy$$

$$= i \int_{y=0}^{1} (1 - y^2 + 2iy)\,dy$$

$$= i\left[y - \frac{y^3}{3} + iy^2\right]_0^1$$

$$= i\left[1 - \frac{1}{3} + i\right] = i\left[\frac{2}{3} + i\right]$$

$$\int_{AB} z^2\,dz = -1 + \frac{2i}{3} \qquad \ldots (2)$$

Along BC, $y = 1$, $\therefore \quad dy = 0 ; \ 1 \le x \le 0$

$$z^2\,dz = (x + iy)^2\,(dx + i\,dy) = (x + i)^2\,dx$$

$$\int_{BC} z^2\,dz = \int_{x=1}^{0} (x^2 + 2ix - 1)\,dx$$

$$= \left[\frac{x^3}{3} + ix^2 - x\right]_{x=1}^{0} = \frac{-1}{3} - i + 1 = \frac{2}{3} - i$$

$$\int_{BC} z^2\,dz = \frac{2}{3} - i \qquad \ldots (3)$$

Along CO, $x = 0 \ \therefore \quad dx = 0, \ 1 \le y \le 0$

$$z^2\,dz = (x+iy)^2\,(dx + i\,dy) = -i\,y^2\,dy$$

$$\int_{CO} z^2\,dz = \int_{y=1}^{0} -i\,y^2\,dy = -i\left[\frac{y^3}{3}\right]_1^0 = \frac{i}{3}$$

$$\int_{CO} z^2\,dz = \frac{i}{3} \qquad \ldots(4)$$

Adding (1), (2), (3), (4) we have,

$$\int_{OA} z^2\,dz + \int_{AB} z^2\,dz + \int_{BC} z^2\,dz + \int_{CO} z^2\,dz$$

$$= \frac{1}{3} + \left(-1 + \frac{2i}{3}\right) + \left(\frac{2}{3} - i\right) + \frac{i}{3} = 0$$

Thus $\int_C z^2\,dz = 0$. **Hence Cauchy's theorem is verified.**

---

22. *Show that* $\int_C |z|^2\,dz = i - 1$ *where C is the square having vertices* $(0,0)(1,0)(1,1)(0,1)$. *Give reason for Cauchy's theom. not being satisfied.*

>> Referring to Problem-15 we have obtained

$$\int_C |z|^2\,dz = i - 1.$$

According to Cauchy's theorem we must have had

$$\int_C |z|^2\,dz = 0.$$

But Cauchy's theorem imposes the condition on $f(z)$ to be analytic.

Here $f(z) = |z|^2$ or $u + iv = x^2 + y^2$

$\therefore \quad u = x^2 + y^2$ and $v = 0$

Also $u_x = 2x,\ u_y = 2y,\ v_x = 0,\ v_y = 0$

Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$ are not satisfied.

Hence, $f(z) = |z|^2$ **is not analytic.**

This is the reason for Cauchy's theorem not being satisfied.

---

23. *Verify Cauchy's theorem for the function $f(z) = z e^{-z}$ over the unit circle with origin as the centre.*

>> We have to evaluate $\int_C z e^{-z}\, dz$ where C is the circle $|z| = 1$.

$\therefore \quad z = e^{i\theta},\ 0 \le \theta \le 2\pi, \quad dz = i e^{i\theta}\, d\theta$

$$\int_C z e^{-z}\, dz = \int_{\theta=0}^{2\pi} e^{i\theta}\, e^{-e^{i\theta}}\, i e^{i\theta}\, d\theta = i \int_{\theta=0}^{2\pi} e^{2i\theta}\, e^{-e^{i\theta}}\, d\theta$$

Put $e^{i\theta} = t \quad \therefore \quad e^{i\theta}\, i\, d\theta = dt$ or $d\theta = \dfrac{dt}{it}$

When $\theta = 0, \quad t = e^0 = 1$

$\theta = 2\pi, \quad t = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$

$$\int_C z e^{-z}\, dz = i \int_{t=1}^{1} t^2\, e^{-t} \frac{dt}{it} = \int_{t=1}^{1} t\, e^{-t}\, dt = 0$$

(Since both the limits are same, the value of the integral is zero)

Thus $\int_C z e^{-z} = 0$. **Hence Cauchy's theorem is verified.**

---

24. *If $f(z) = u + iv$ where u and v are functions of x, y having continuous partial derivatives in a region R and $\int_C f(z)\, dz = 0$ for every simple closed curve C in the region then show that $f(z)$ is analytic in the region R.*

Proof :

We have, $\int_C f(z)\, dz = \int_C (u + iv)(dx + i dy)$

i.e., $$\int_C f(z)\, dz = \int_C (u\, dx - v\, dy) + i \int_C (v\, dx + u\, dy) \quad \ldots (1)$$

We also have Green's theorem in a plane,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Applying this theorem to the line integrals in the RHS of (1) we obtain

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

As $\int_C f(z)\,dz = 0$ by data, both the terms in the RHS must be zero.

Hence we must have $-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ and $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$

$\therefore \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ at all points in R.

These are Cauchy-Riemann equations and hence we can say that $f(z) = u + iv$ satisfy Cauchy-Riemann equations in R. Since we also have by data that the partial derivatives are continuous, we conclude that $f(z)$ is analytic in the region R.

Remark : *This is the converse of Cauchy's theorem and is known as Morera's theorem.*

Problems on Cauchy's integral formula

Working procedure for problems

- We need to evaluate integrals of the form $\int_C \frac{f(z)}{(z-a)}\,dz$; $\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$ over a given closed curve C.
- Firstly we have to find out whether the point $z = a$ lies inside or outside the given curve C. [*Refer to the Note in Problem-25*]
- If $z = a$ is inside C then we use Cauchy's integral formula in the forms

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

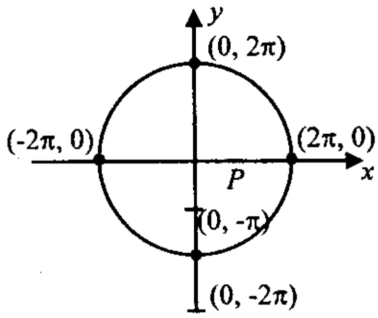
- If the point $z = a$ is outside C and supposing that $F(z) = \frac{f(z)}{(z-a)}$ or $\frac{f(z)}{(z-a)^{n+1}}$ is analytic inside and on the given curve C we can conclude that $\int_C F(z)\,dz = 0$ by Cauchy's theorem.
- In other words if $z = a$ is outside C the value of the integral is zero.

25. *Evaluate* $\int_C \frac{e^z}{z+i\pi}\, dz$ *over each of the following contours C :*

(a) $|z| = 2\pi$ *(b)* $|z| = \pi/2$ *(c)* $|z-1| = 1$

>> We have to evaluate the integral which can be written in the form $\int_C \frac{e^z}{z-(-i\pi)}\, dz$ which is of the form $\int_C \frac{f(z)}{z-a}\, dz$

Here $f(z) = e^z,\ a = -i\pi$



(a) $|z| = 2\pi$ is a circle with centre origin and radius 2π

The point $z = a = -i\pi$ is the point $P(0, -\pi)$ lies within the circle $|z| = 2\pi$

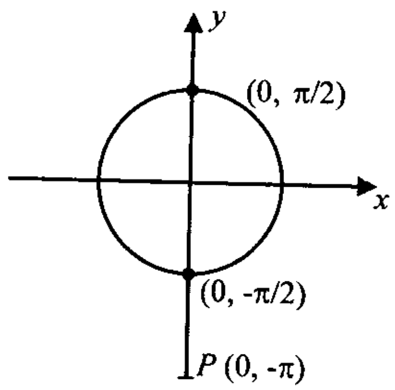
We have Cauchy's integral formula $\int_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$

We have, $f(z) = e^z,\ a = -i\pi$

$$\therefore \quad \int_C \frac{e^z}{z+i\pi}\, dz = 2\pi i f(-i\pi) = 2\pi i e^{-i\pi} = 2\pi i\,(\cos\pi - i\sin\pi) = -2\pi i$$

Thus $\int_C \frac{e^z}{z+i\pi}\, dz = -2\pi i$, where C is the circle $|z| = 2\pi$

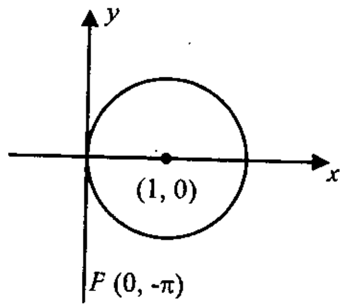
(b) $|z| = \pi/2$ is a circle with centre origin and radius $\pi/2$. The point $P(0, -\pi)$ lies outside the circle $|z| = \pi/2$ and $\frac{e^z}{z+i\pi}$ is analytic inside and on the circle $|z| = \pi/2$.



By Cauchy's theorem,

$$\int_C \frac{e^z}{z+i\pi}\, dz = 0, \text{ where } C: \, | z | = \pi/2.$$

(c) $| z-1 | = 1$ is a circle with centre at $z = a = 1$ and radius 1. That is a circle with centre $(1, 0)$ and radius 1.



The point $P(0, -\pi)$ lies outside the circle $| z-1 | = 1$ and hence by Cauchy's theorem

$$\int_C \frac{e^z}{z+i\pi}\, dz = 0, \text{ where } C \text{ is } | z-1 | = 1$$

Note : *In order to decide whether a point lies inside or outside a given circle we can employ the following procedure.*

If A is the centre and r is the radius of the given circle, a point P lies inside the circle if the distance AP is less than r, outside the circle if the distance AP is greater than r. If $AP = r$, obviously P is a point on the circle.

In this problem we have

Case-(i) : $A = (0, 0)$, $P = (0, -\pi)$; $r = 2\pi$

$AP = \pi < 2\pi \Rightarrow P$ is inside the circle.

Case-(ii) : $A = (0, 0)$, $P = (0, -\pi)$; $r = \pi/2$

$AP = \pi > \pi/2 \Rightarrow P$ is outside the circle.

Case-(iii) : $A = (1, 0)$, $P = (0, -\pi)$; $r = 1$

$AP = \sqrt{1+\pi^2} > 1 \Rightarrow P$ is outside the circle.

---

26. *Evaluate* $\int_C \frac{dz}{z^2-4}$ *over the following curves* C.

(a) $C : |z| = 1$ *(b)* $C : |z| = 3$ *(c)* $C : |z+2| = 1$

>> Consider $\frac{1}{z^2-4} = \frac{1}{(z-2)(z+2)}$

Resolving into partial fractions,

$$\frac{1}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$$

or $1 = A(z+2) + B(z-2)$

Putting $z = 2$: $1 = A(4)$ $\therefore$ $A = 1/4$

$z = -2$: $1 = B(-4)$ $\therefore$ $B = -1/4$

Now $\frac{1}{(z-2)(z+2)} = \frac{1}{4} \cdot \frac{1}{z-2} - \frac{1}{4} \cdot \frac{1}{z+2}$

$$\therefore \quad \int_C \frac{dz}{(z-2)(z+2)} = \frac{1}{4}\int_C \frac{dz}{z-2} - \frac{1}{4}\int_C \frac{dz}{z-(-2)} \quad \ldots (1)$$

(a) $C : |z| = 1$; $z = a = 2$ and $z = a = -2$ both of them lie outside C.

Thus by Cauchy's theorem $\int_C \frac{dz}{z^2-4} = 0$ where $C : |z| = 1$

(b) $C : |z| = 3$; $z = a = 2$ and $z = a = -2$ lies inside the circle. Also in each of the integrals as in the RHS of (1), $f(z) = 1$.

Applying Cauchy's integral formula,

$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$ we obtain

$$\int_C \frac{dz}{z-2} = 2\pi i f(2) = 2\pi i . 1 = 2\pi i$$

$$\int_C \frac{dz}{z+2} = 2\pi i f(-2) = 2\pi i \cdot 1 = 2\pi i$$

Substituting these in the RHS of (1) we have,

$$\int_C \frac{dz}{z^2-4} = \frac{1}{4}(2\pi i) - \frac{1}{4}(2\pi i) = 0$$

Thus $\int_C \frac{dz}{z^2-4} = 0$ where $C : |z| = 3$

(c) $C : |z+2| = 1$. This is a circle with centre $(-2, 0)$ and radius 1.

Let $A = (-2, 0)$ and $P = (2, 0)$ Hence $AP = \sqrt{4} = 2 > 1$

$\therefore$ the point $z = a = 2$ lies outside the circle and clearly the point $z = a = -2$ being $(-2, 0)$ lies inside the circle.

Hence by Cauchy's theorem $\int_C \frac{dz}{z-2} = 0$

Also by Cauchy's integral formula,

$$\int_C \frac{dz}{z+2} = \int_C \frac{dz}{z-(-2)} = 2\pi i\, f(-2) \text{ where } f(z) = 1$$

$$\therefore \quad \int_C \frac{dz}{z+2} = 2\pi i \cdot 1 = 2\pi i$$

Substituting these value in the RHS of (1) we have,

$$\int_C \frac{dz}{z^2-4} = \frac{1}{4} \cdot 0 - \frac{1}{4} \cdot 2\pi i = \frac{-\pi i}{2}$$

Thus $\int_C \frac{dz}{z^2-4} = -\frac{\pi i}{2}$ where $C : |z+2| = 1$

---

27. *Evaluate* $\int_C \frac{e^z}{z-i\pi}$ *where C is the circle* ***(a)*** $|z| = 2\pi$ ***(b)*** $|z| = \pi/2$

>> This problem is similar to Problem - 25..

In the case (a), the point $z = i\pi$ lies inside the circle $|z| = 2\pi$.

We have Cauchy's integral formula

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$

Taking $f(z) = e^z$ and $a = i\pi$

$$\int_C \frac{e^z}{z-i\pi}\, dz = 2\pi i\, f(i\pi) = 2\pi i e^{i\pi} = 2\pi i(\cos\pi + i\sin\pi) = -2\pi i$$

Thus $\int_C \frac{e^z}{z-i\pi}\, dz = -2\pi i$ for $C : |z| = 2\pi$

In the case (b) the point $z = i\pi$ lies outside the circle $|z| = \pi/2$, $\phi(z) = \frac{e^z}{z-i\pi}$ is analytic inside and on this circle.

Hence by Cauchy's theorem $\int_C \phi(z)\, dz = 0.$

Thus $\int_C \frac{e^z}{z-i\pi}\, dz = 0$ for $C : |z| = \pi/2.$

---

28. *Evaluate* $\int_C \frac{e^{2z}}{(z+1)(z-2)}\, dz$ *where C is the circle* $|z| = 3.$

>> The points $z = a = -1$, $z = a = 2$ being $(-1, 0)$ $(2, 0)$ lies inside $|z| = 3.$

Now we shall resolve $\frac{1}{(z+1)(z-2)}$ into partial fractions.

Let $\frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$

or $1 = A(z-2) + B(z+1)$

Putting $z = 2$, $1 = B(3)$ $\therefore$ $B = 1/3$

Putting $z = -1$, $1 = A(-3)$ $\therefore$ $A = -1/3$

Hence $\frac{1}{(z+1)(z-2)} = \frac{-1}{3} \cdot \frac{1}{z+1} + \frac{1}{3} \cdot \frac{1}{z-2}$

$$\therefore \quad \frac{e^{2z}}{(z+1)(z-2)} = \frac{1}{3}\left[\frac{e^{2z}}{z-2} - \frac{e^{2z}}{z+1}\right]$$

$$\Rightarrow \quad \int_C \frac{e^{2z}\, dz}{(z+1)(z-2)} = \frac{1}{3}\left[\int_C \frac{e^{2z}}{z-2}\, dz - \int_C \frac{e^{2z}}{z+1}\, dz\right] \quad \ldots (1)$$

We have Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Taking $f(z) = e^{2z}$ and $a = 2, -1$ respectively we obtain

$$\int_C \frac{e^{2z}}{z-2}\,dz = 2\pi i f(2) = 2\pi i e^4$$

and $$\int_C \frac{e^{2z}}{z+1}\,dz = 2\pi i f(-1) = 2\pi i e^{-2} = \frac{2\pi i}{e^2}$$

Substituting these in the RHS of (1) we obtain,

$$\int_C \frac{e^{2z}\,dz}{(z+1)(z-2)} = \frac{1}{3}\left[2\pi i e^4 - \frac{2\pi i}{e^2}\right]$$

Thus $$\int_C \frac{e^{2z}\,dz}{(z+1)(z-2)} = \frac{2\pi i}{3}\left[e^4 - \frac{1}{e^2}\right]$$

---

29. *Evaluate* $\int_C \frac{e^{3z}}{z^2}\,dz$ *over* $C : |z| = 1$.

>> The point $z = 0$ lies within the circle $|z| = 1$ and we have Cauchy's integral formula in the generalised form,

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Taking $f(z) = e^{3z}$, $a = 0$, $n = 1$ in this formula we obtain,

$$\int_C \frac{e^{3z}}{z^2}\,dz = \frac{2\pi i}{1!} f'(a) \; ; \quad \text{Also } f'(z) = 3e^{3z}$$

$$\therefore \quad \int_C \frac{e^{3z}}{z^2}\,dz = 2\pi i\,(3e^0) = 2\pi i(3) = 6\pi i$$

Thus $$\int_C \frac{e^{3z}}{z^2}\,dz = 6\pi i.$$

---

30. *Evaluate* $\int_C \frac{z^2+z+1}{(z-2)^3}\, dz$ *over* $C : |z| = 3$

>> The point $z = 2$ lies inside the circle $|z| = 3$

We have generalised Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Taking $f(z) = z^2 + z + 1$, we obtain $f''(z) = 2 \quad \therefore \quad f''(2) = 2$

Also by taking $a = 2,\ n = 2$ we have

$$\int_C \frac{z^2 + z + 1}{(z-2)^3}\, dz = \frac{2\pi i}{2!} f''(2) = \frac{2\pi i}{2} \cdot 2 = 2\pi i$$

Thus $\int_C \frac{z^2 + z + 1}{(z-2)^3}\, dz = 2\pi i.$

---

31. *Evaluate* $\int_C \frac{e^{\pi z}}{(2z-i)^3}\, dz$ *where C is the circle* $|z| = 1.$

>> We can write the given integral in the form

$$\int_C \frac{e^{\pi z}}{[2(z-i/2)]^3} = \frac{1}{8} \int_C \frac{e^{\pi z}}{(z-i/2)^3}$$

The point $z = i/2$ being $(0, 1/2)$ lies within the circle $|z| = 1.$

We have generalised Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Taking $f(z) = e^{\pi z},\ a = i/2,\ n = 2$ we have

$$\int_C \frac{e^{\pi z}}{(z-i/2)^3}\, dz = \frac{2\pi i}{2!} f''(i/2) = \pi i f''(i/2)$$

Multiplying by $1/8$ we have,

$$\frac{1}{8} \int_C \frac{e^{\pi z}}{(z-i/2)^3}\, dz = \frac{1}{8} \cdot \pi i f''(i/2) \quad ; \quad \text{But} \quad f''(z) = \pi^2 e^{\pi z}$$